Inverted pendulum

Equations of Motion.

Inverted pendulum is a two degrees of freedom (2DOF) problem.

An input force is applied to a cart.

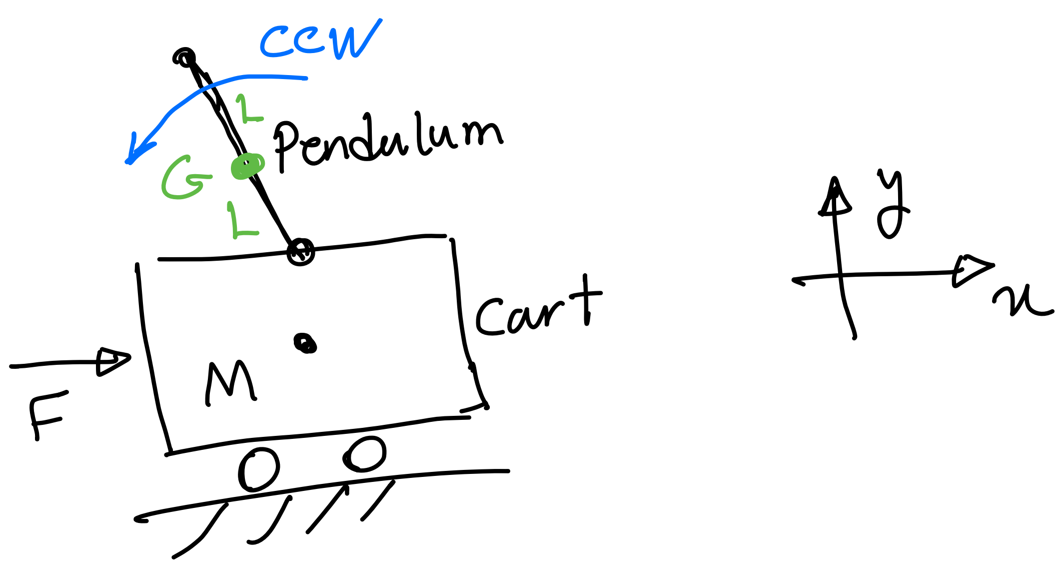
The cart can only translate along x-

The pendulum can rotate and translate

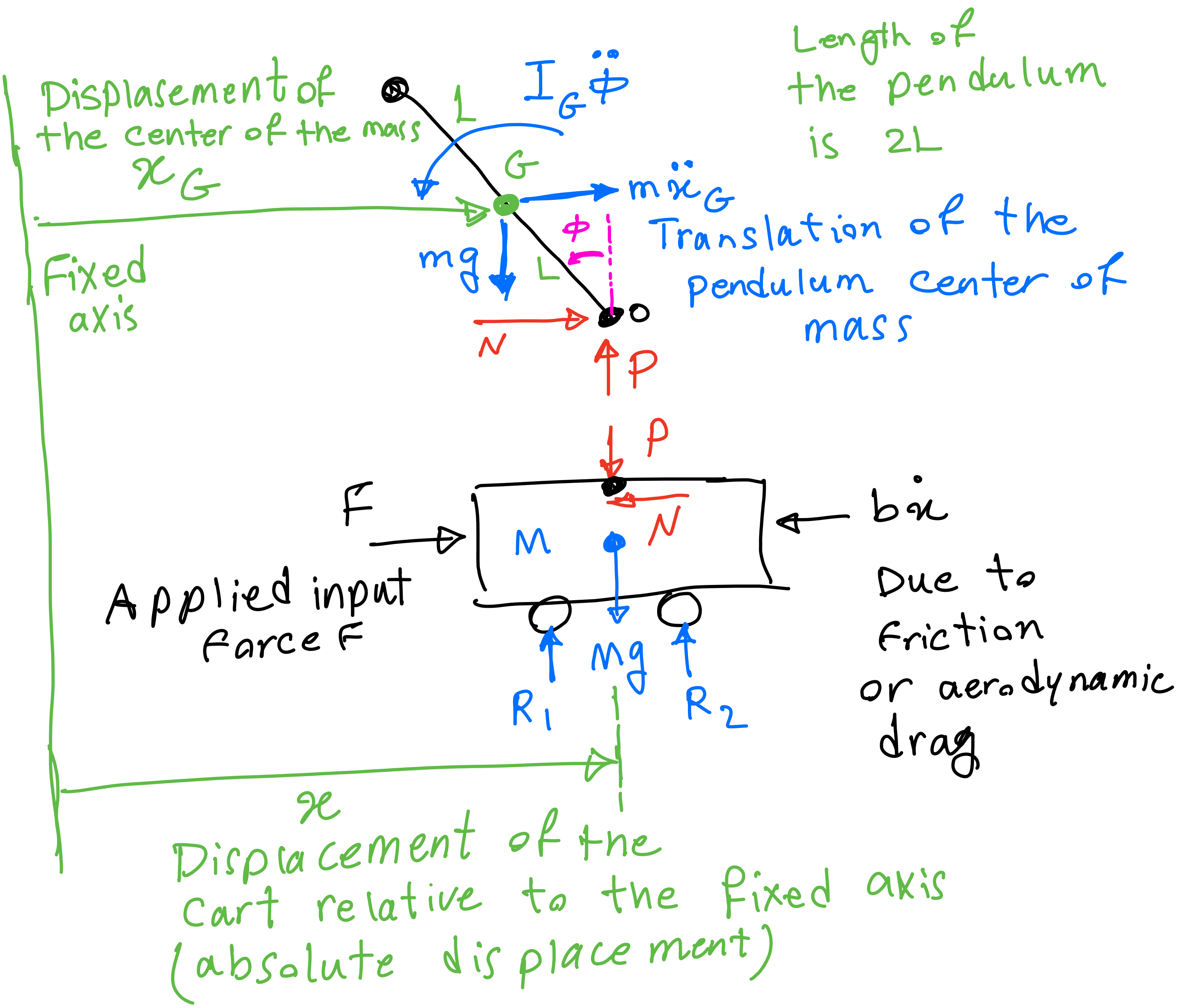
Note: In order to draw the free-body diagram, assume that the pendulum is vertical in equilibrium position initially.

Then, apply force F to the cart and make a prediction for the direction of rotation of the pendulum relative to the cart. This allows to find the correct direction of the motion of the pendulum and therefore the correct FBD as well as correct signs for the equations of motion.

Therefore, if force F is applied to the right we can expect the rotation of the pendulum counter clockwise as in the figure below.



Free-body-diagram (F.B.D.)



Equations of Motion:

The cart:

$$\xrightarrow{+} \sum F_x = M\ddot{x}$$

$$F - b\dot{x} - N = M\ddot{x} \qquad (1)$$

The pendulum:

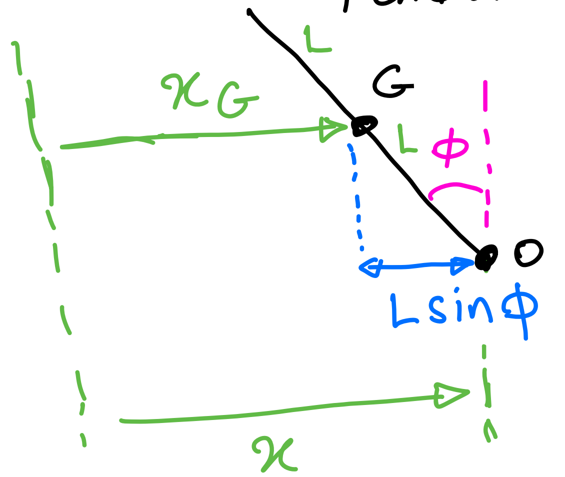
Translation:
(From FBD)

$$\xrightarrow{+} \sum F_x = m\ddot{x}_G$$

$$N = m\ddot{x}_G \qquad (2)$$

x_G in terms of x displacement:

Pendulum



$$x_G = x - L\sin\phi$$

$$\dot{x}_G = \frac{dx}{dt} - L\frac{d}{dt}\sin\phi$$

$$\dot{x}_G = \dot{x} - L\cos\phi\,\dot{\phi}$$

Chain rule:

$$\frac{d}{dt}\sin\phi = \frac{d\sin\phi}{dt}\,\frac{d\phi}{d\phi}$$

$$= \frac{d\sin\phi}{d\phi}\,\frac{d\phi}{dt}$$

$$= \cos\phi\,\dot{\phi}$$

$$\ddot{x}_G = \ddot{x} - L\frac{d}{dt}(\cos\phi\,\dot{\phi})$$

or

$$\ddot{x}_G = \ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right] \qquad (3)$$

product rule:

$$\frac{d}{dt}(\cos\phi\,\dot{\phi}) = \frac{d\cos\phi}{dt}\,\dot{\phi} + \cos\phi\,\frac{d}{dt}\dot{\phi}$$

$$= \frac{d\cos\phi}{d\phi}\,\frac{d\phi}{dt}\,\dot{\phi} + \cos\phi\,\ddot{\phi}$$

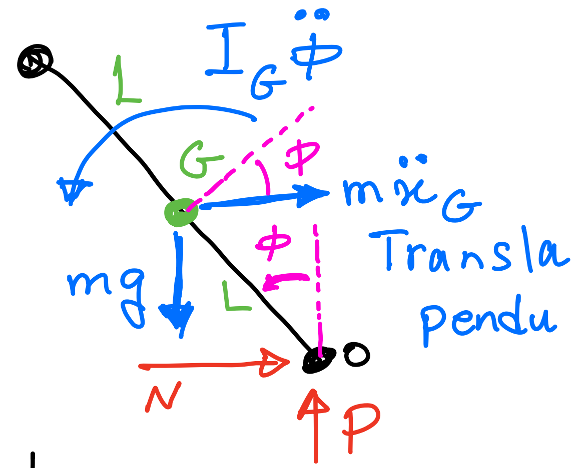
$$= -\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}$$

② & ③ $\Longrightarrow$

$$N = m\ddot{x}_G$$

$$N = m\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\} \quad (4)$$

The pendulum rotation:
(from F.B.D)



$$\overset{+}{\circlearrowleft}\ \Sigma M_o = I_G\ddot{\phi} - m\ddot{x}_G\cos\phi\, L$$

$$\left\{\begin{array}{l} mgL\sin\phi = I_G\ddot{\phi} - m\cos\phi\, L\,\ddot{x}_G \\ (3) \rightarrow \ddot{x}_G = \ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right] \end{array}\right.$$

$$\Longrightarrow mgL\sin\phi = I_G\ddot{\phi} - mL\cos\phi\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\} \quad (5)$$

In a control problem we desire to have small deviation of ϕ.

For small ϕ we have: $\begin{cases} \sin\phi = \phi \\ \cos\phi = 1 \\ \dot{\phi}^2 \cong 0 \end{cases}$

These will linearized the equations $\longrightarrow$

substitute in 5 $\Rightarrow$

$$mgL\phi = I_G\ddot{\phi} - mL\left[\ddot{x} - L\ddot{\phi}\right]$$

$$\Rightarrow \quad -I_G\ddot{\phi} - mL^2\ddot{\phi} + mgL\phi = -mL\ddot{x}$$

or

$$(I_G + mL^2)\ddot{\phi} - mgL\phi = mL\ddot{x} \qquad a$$

1 & 4 $\Rightarrow$

$$F - b\dot{x} - N = M\ddot{x} \qquad 1$$

$$N = m\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\} \qquad 4$$

Small ϕ

$\begin{cases} \sin\phi = \phi \\ \cos\phi = 1 \\ \dot{\phi}^2 = 0 \end{cases}$

Linearizing

$$\rightarrow F - b\dot{x} - m\left[\ddot{x} - L\ddot{\phi}\right] = M\ddot{x}$$

$$F - b\dot{x} - m\ddot{x} + mL\ddot{\phi} = M\ddot{x}$$

Rearranging: $\quad -(M+m)\ddot{x} - b\dot{x} + mL\ddot{\phi} = -F$

or

$$(M+m)\ddot{x} + b\dot{x} - mL\ddot{\phi} = F \quad (b)$$

Therefor, the Equations of Motion are:

(a) & (b) →

$$\begin{cases} (I_G + mL^2)\ddot{\phi} - mgL\phi = mL\ddot{x} \\ (M+m)\ddot{x} + b\dot{x} - mL\ddot{\phi} = F \end{cases}$$

Next, we will find the transfer functions:

$$\frac{\Phi(s)}{F(s)} \quad \text{and} \quad \frac{X(s)}{F(s)}$$

Also, the equations of Motion can be written in state-space representation

$$[\dot{x}] = [A][x] + [B][u]$$

$$[Y] = [C][x] + [D][u]$$

or

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = [A] \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + [B]\, F$$

$$[Y] = [C] \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + [D]\, F$$