Instrumentation and Controls

ETM 3301

Lecture 9

Instructor

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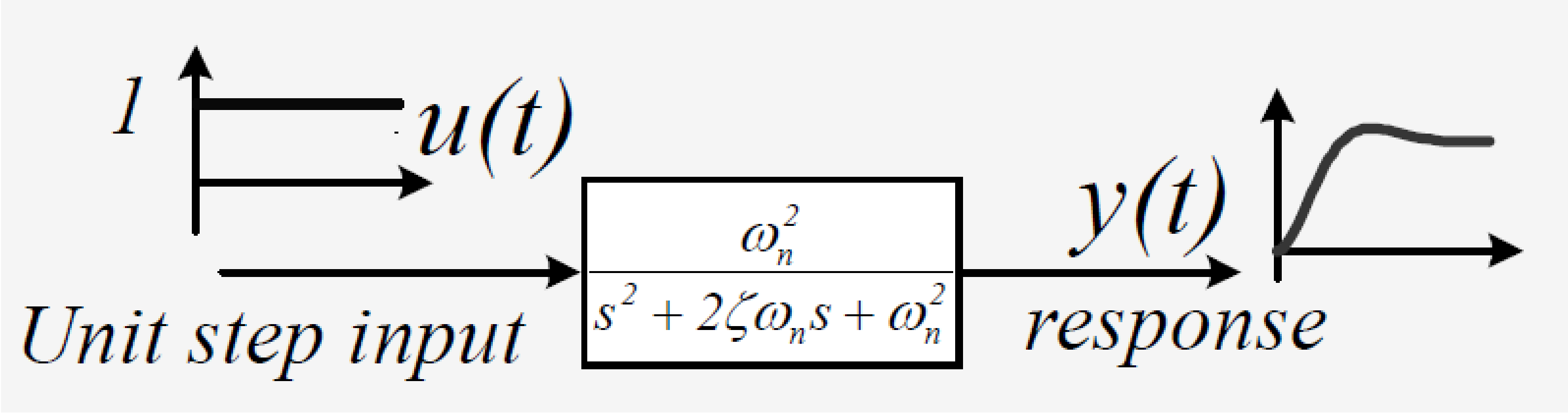
Unit step response of a 2nd order system



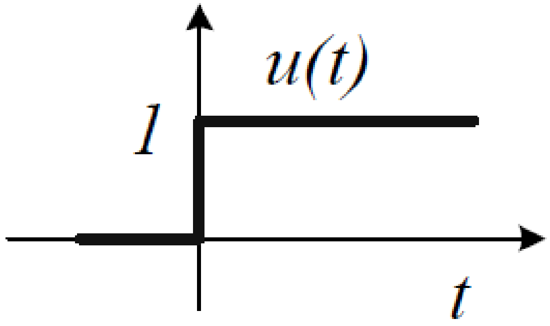
- For the underdamped case (0<ζ<1), the unit step response for a second order system is:

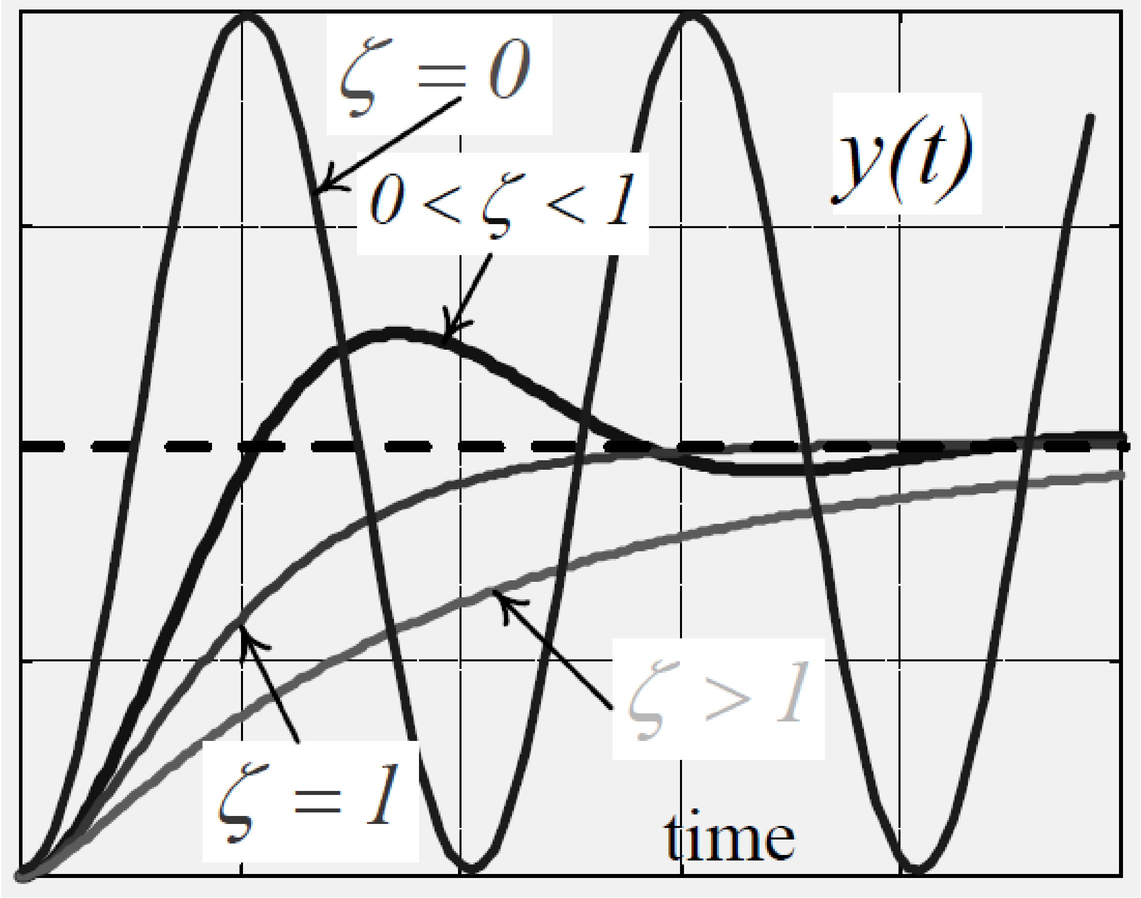
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

where $\phi = \cos^{-1}\zeta \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

Second Order System Step Responses

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

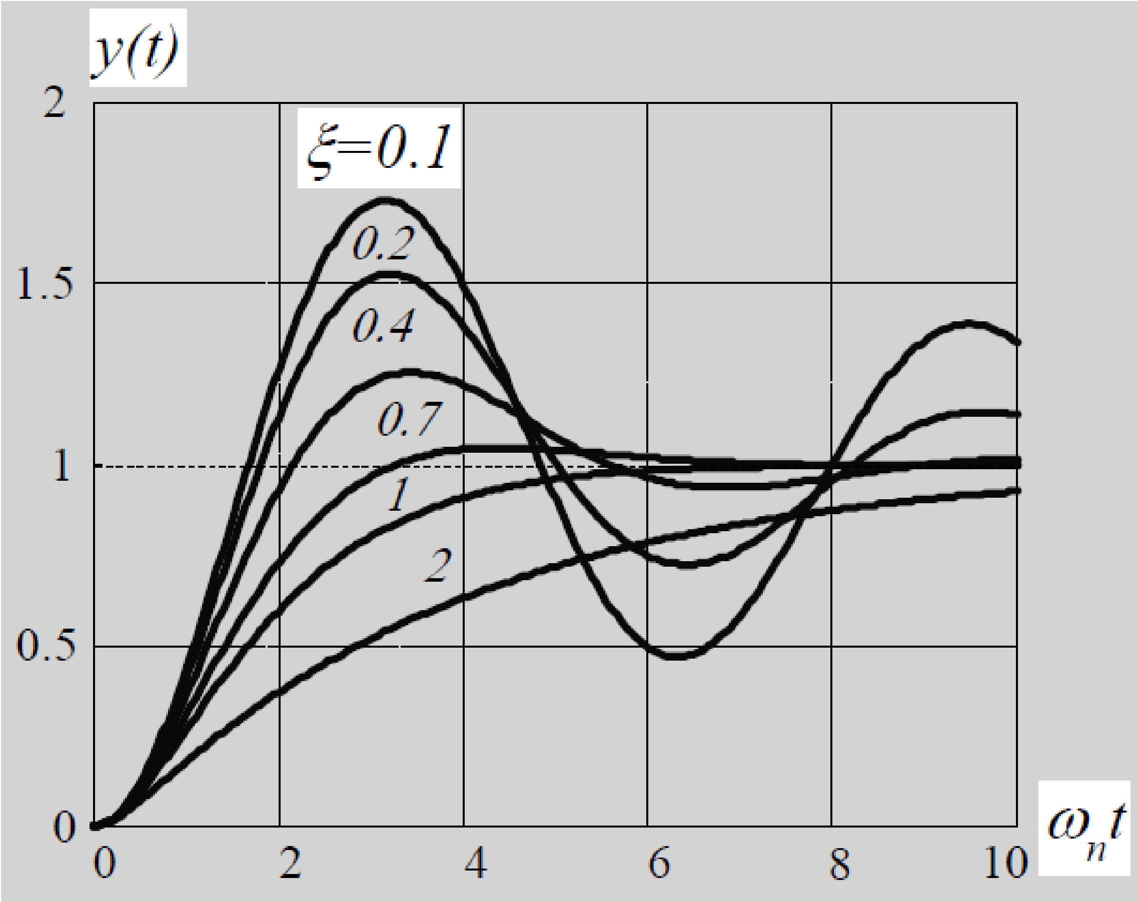


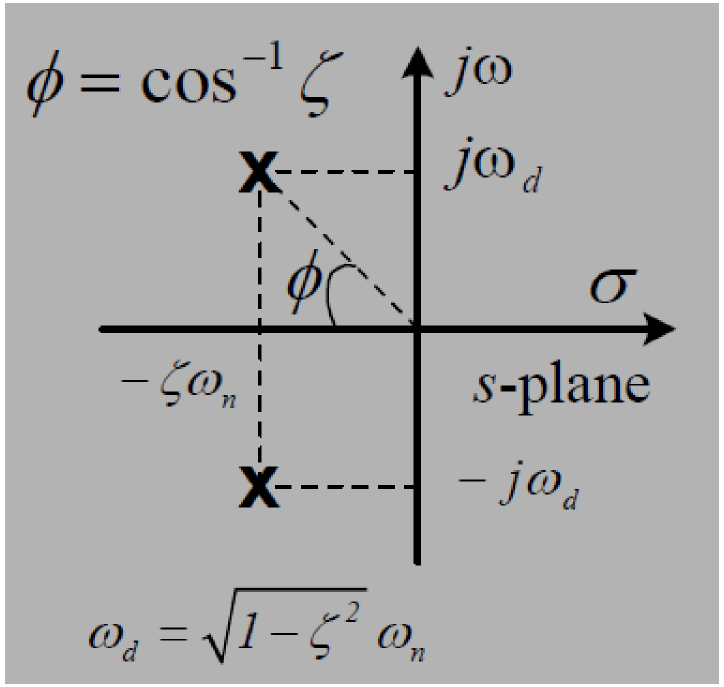


2^{nd} order system pole position and response

- The unit step responses for various values of ζ:

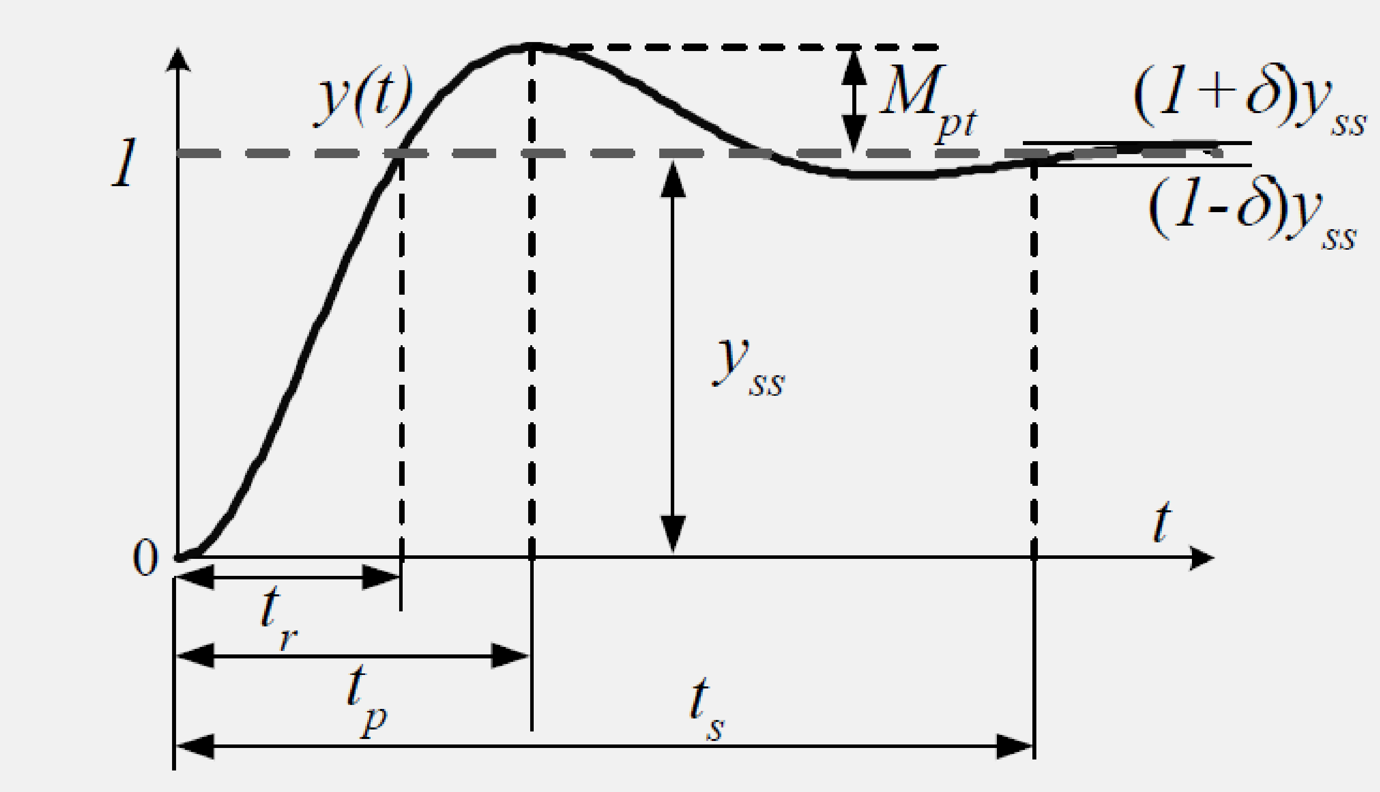
$$s_{1,2} = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$$
$$= -\zeta\omega_n \pm j\,\omega_d$$





Pole positions

2^{nd} order system unit step response characteri-stics



t_r	*rise time*
t_p	*time to first peak*
M_{pt}	*overshoot*
t_s	*settling time*

Four performance measures!

the response to fall within a prescribed band about the final steady value

Time To First Peak (maximum value)

- The time to the first peak t_p can be found by differentiating *y(t)* and setting the derivative equal to zero.

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$\frac{dy(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t = 0$$

$$\Rightarrow \omega_n \sqrt{1-\zeta^2}\, t = \pi$$

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$$\Rightarrow \quad t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Time To First Peak (maximum value)

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$\frac{dy(t)}{dt} = -\frac{1}{\sqrt{1-\zeta^2}} \left(\frac{de^{-\zeta\omega_n t}}{dt} \sin(\omega_d t + \phi) + e^{-\zeta\omega_n t} \frac{d \sin(\omega_d t + \phi)}{dt} \right)$$

$$= -\frac{1}{\sqrt{1-\zeta^2}} \left(-\zeta\omega_n e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + e^{-\zeta\omega_n t} \omega_d \cos(\omega_d t + \phi) \right)$$

$$= -\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left(-\zeta \sin(\omega_d t + \phi) + \sqrt{1-\zeta^2} \cos(\omega_d t + \phi) \right)$$

$$= -\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left(-\cos\phi \sin(\omega_d t + \phi) + \sin\phi \cos(\omega_d t + \phi) \right)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi - \phi) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t)$$

Overshoot and Percentage Overshoot

- At the first peak, the response (maximum value) is:

$$y(t_p) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \sin(\omega_d \frac{\pi}{\omega_d} + \phi) = 1 + \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

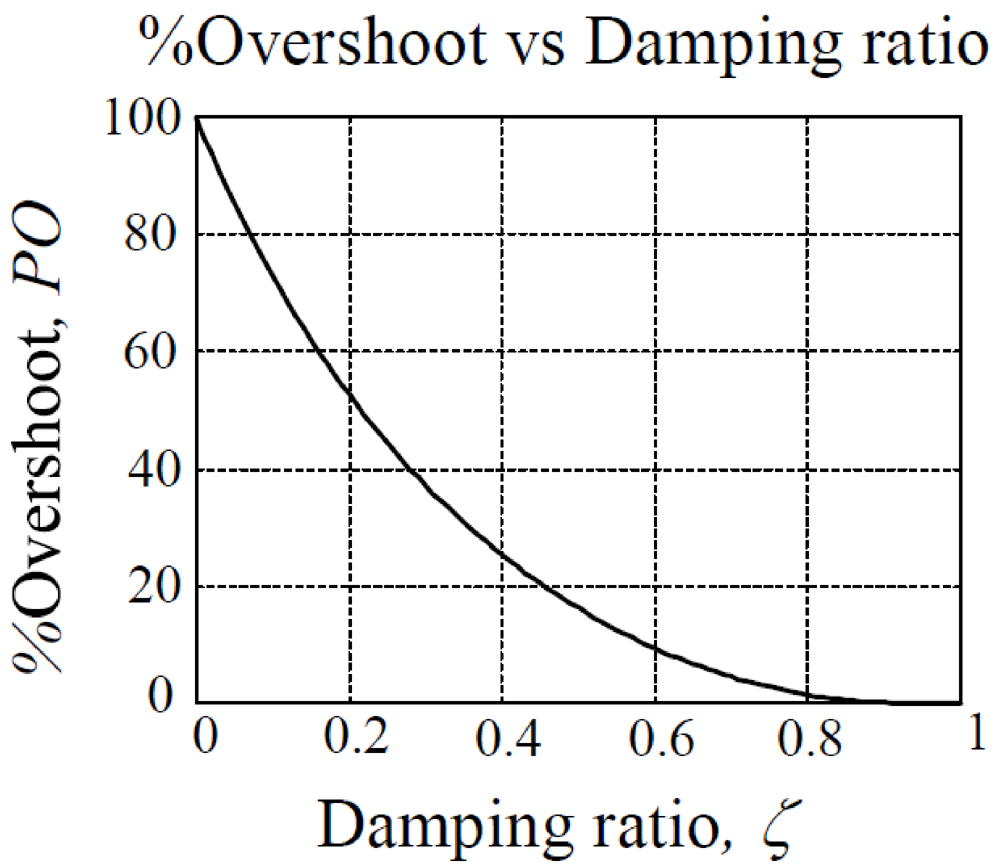
- Overshoot is:

$$M_{pt} = y(t_p) - y_{ss} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

- Percentage Overshoot (More useful measure):

$$PO = \frac{M_{pt}}{y_{ss}} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

Percentage Overshoot vs Damping Ratio



$$PO = \frac{M_{pt}}{y_{ss}} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

Rise Time

- The rise time t_r is found by setting $y(t) = 1$ and solving for the lowest value of t, this gives

$$y(t_r) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_r} \sin(\omega_n \sqrt{1-\zeta^2}\, t_r + \phi) = 1$$

$$\Rightarrow \sin(\omega_n \sqrt{1-\zeta^2}\, t_r + \phi) = 0$$

$$\Rightarrow \omega_n \sqrt{1-\zeta^2}\, t_r + \phi = \pi$$

$$\Rightarrow \quad t_r = \frac{\pi - \phi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi - \cos^{-1}\zeta}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi - \phi}{\omega_d}$$

Settling Time

- The setting time is the time required for an output to reach and remain within a given error band (e.g. 2% or 5%) following a step input.

$$u(t) = 1$$

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$e(t) = u(t) - y(t) = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$|e(t)| \le e^{-\zeta\omega_n t_s} = \delta \leftarrow \text{error band}$$

Settling Time

$$|e(t)| \leq e^{-\zeta\omega_n t} = \delta \leftarrow \text{error band} \qquad \zeta\omega_n t \geq \ln\delta$$

- The setting time depends on error band and can be estimated approximately.

$$t_s = \frac{\ln\delta}{\zeta\omega_n}$$

2% error band: $t_s \approx \dfrac{4}{\zeta\omega_n}$

5% error band: $t_s \approx \dfrac{3}{\zeta\omega_n}$

2nd order system example

$$G(s) = \frac{5}{s^2 + 2s + 5}; \quad \omega_n = 2.2361; \quad \zeta = 0.4472$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.57\,\text{sec}$$

$$M_{pt} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.118$$

$$y(t) = 1 - 1.118e^{-t}\sin(2t + 63.4^o)$$

$$y_{ss} = 1$$

$$PO = \frac{M_{pt}}{y_{ss}} = 11.8\%$$

2% error band:

$$t_s = \frac{4}{\zeta\omega_n} = 4\,\text{sec}$$

5% error band:

$$t_s = \frac{4}{\zeta\omega_n} = 3\,\text{sec}$$

2^{nd} Order System Step Response Performance Measures

PO	*Percentage overshoot*
t_s	*Settling time*
t_r	*Rise time*
t_p	*Time to first peak*

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

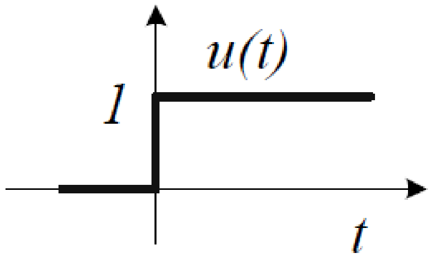
$$t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

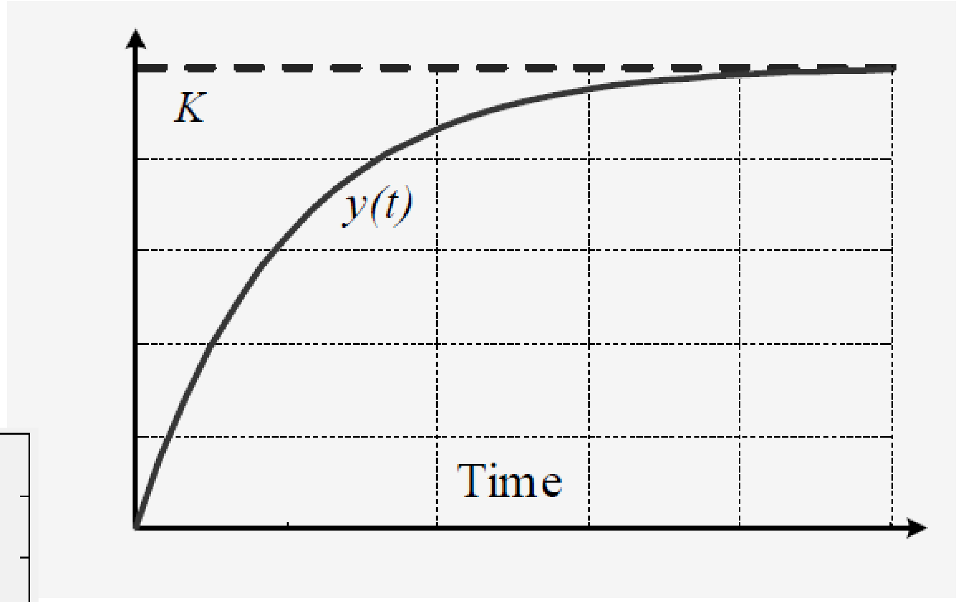
$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)\times 100\%$$

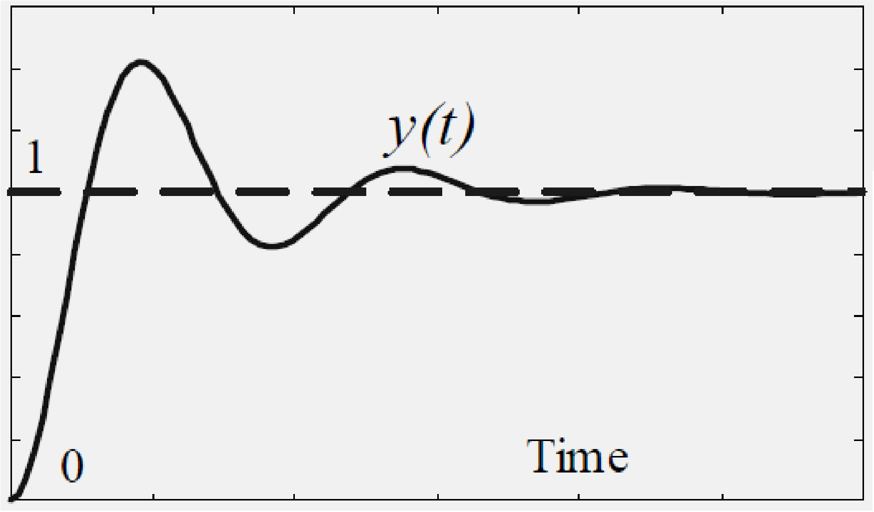
$$t_s = \begin{cases} \dfrac{3}{\zeta\omega_n} & \text{5\% error band} \\ \dfrac{4}{\zeta\omega_n} & \text{2\% error band} \end{cases}$$

First and Second Order System Responses

- Responses of almost all dynamic systems are similar to either 1st or 2nd order system response!



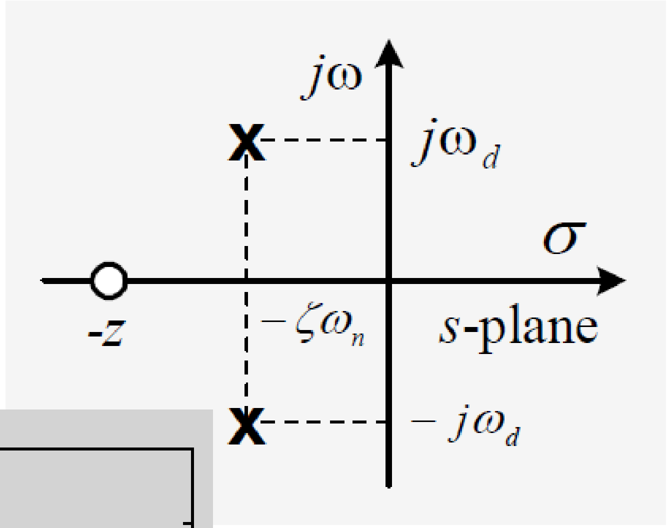


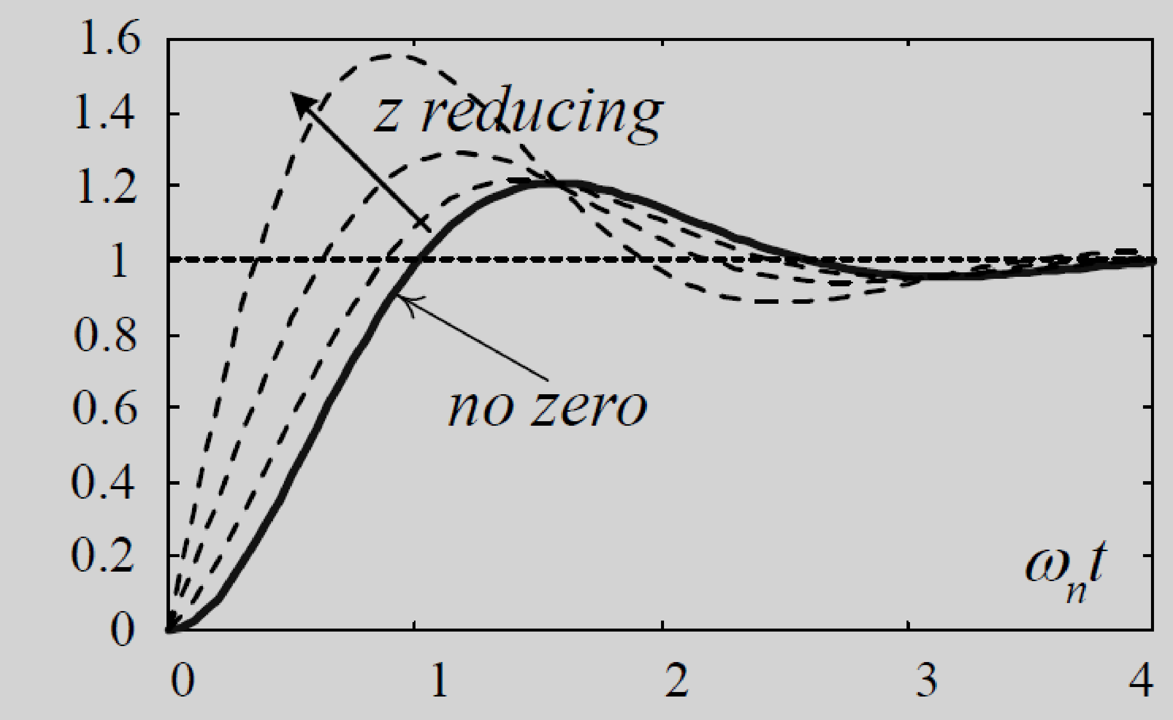


Second Order + Additional Zero

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{s+z}{z}\right)$$

$$0 < \zeta < 1, \quad z > 0$$





Second Order + Additional Pole

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{a}{s+a}\right)$$

$$0 < \zeta < 1, \quad a > 0$$