Instrumentation and Controls

ETM 3301

Lecture 8

Instructor

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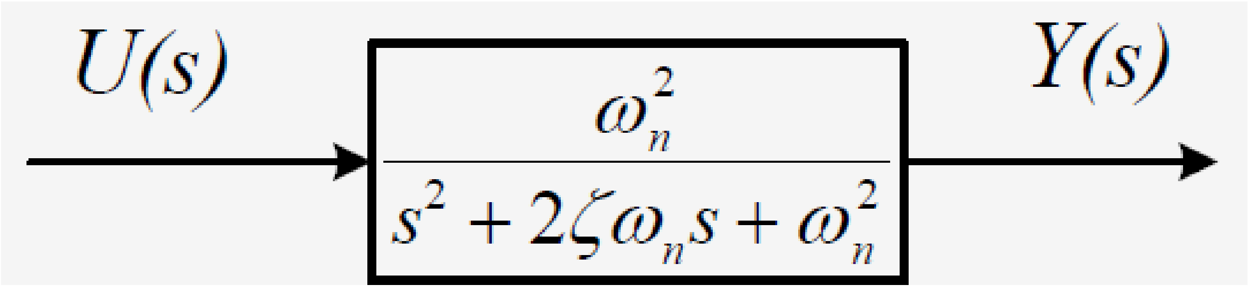
Second Order System TF

- A 2nd order system transfer function (TF) is:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ *damping ratio*

ω_n *undamped natural frequency*



Pole Locations of second order system, 1

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Characteristic equation (CE): $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$$s^2 + 2\zeta\omega_n s + (\zeta\omega_n)^2 - (\zeta\omega_n)^2 + \omega_n^2 = 0$$

$$\Rightarrow (s + \zeta\omega_n)^2 - (\zeta^2 - 1)\omega_n^2 = 0$$

$$\Rightarrow (s + \zeta\omega_n)^2 = (\zeta^2 - 1)\omega_n^2$$

Case 1: $\zeta > 1$ $\qquad$ $s_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$

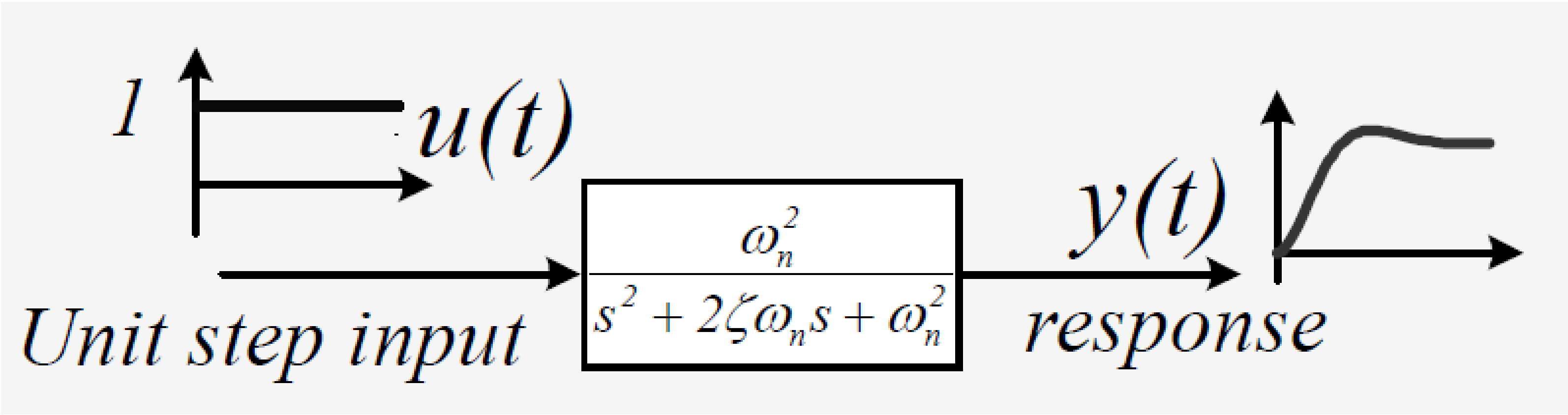
Overdamped $\qquad$ two real distinct poles

Pole Locations of second order system, 2

$$\text{CE}: \quad (s+\zeta\omega_n)^2 = (\zeta^2 - 1)\omega_n^2$$

Case 2:	$\zeta = 1$	$s_{1,2} = -\zeta\omega_n$
Critically damped		two real repeated poles
Case 3:	$\zeta < 1$	$s_{1,2} = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$
Underdamped		two complex conjugate poles
Case 4:	$\zeta = 0$	$s_{1,2} = \pm j\,\omega_n$
Undamped		two imaginary poles

Unit step response of a 2nd order system



- For the underdamped case ($0<\zeta<1$), the unit step response for a second order system is:

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

where $\phi = \cos^{-1}\zeta \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

Unit step response of a 2nd order system

$$Y(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{\omega_n^2 + s^2 + 2\zeta\omega_n s - (s^2 + 2\zeta\omega_n s)}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2\zeta\omega_n s + (\zeta\omega_n)^2 - (\zeta\omega_n)^2 + \omega_n^2$$

$$= (s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) = (s + \zeta\omega_n)^2 + \omega_d^2$$

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Unit step response of a 2nd order system

$$Y(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\frac{\omega_d}{\omega_d}$$

$$=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Inverse Laplace Transform

$$y(t)=1-e^{-\zeta\omega_n t}\cos(\omega_d t)-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_d t)+\zeta\sin(\omega_d t)\right)$$

Unit step response of a 2nd order system

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right)$$

Define: $\phi = \cos^{-1}\zeta$ Then: $\zeta = \cos\phi, \quad \sqrt{1-\zeta^2} = \sin\phi$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\phi\cos(\omega_d t) + \cos\phi\sin(\omega_d t)\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \phi\right)$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

2^{nd} order system example

$$G(s) = \frac{5}{s^2 + 2s + 5} \Leftrightarrow \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 5 \Rightarrow \omega_n = \sqrt{5} = 2.2361$$

$$2\zeta\omega_n = 2 \Rightarrow \zeta = \frac{2}{2\omega_n} = \frac{1}{\sqrt{5}} = 0.4472 < 1$$

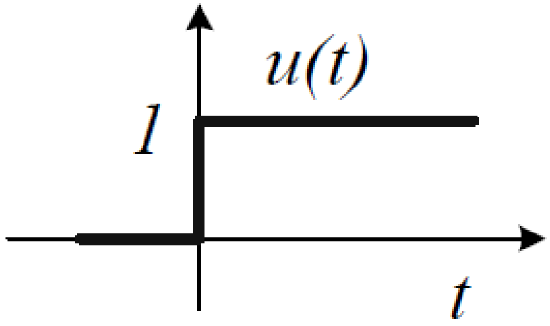
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

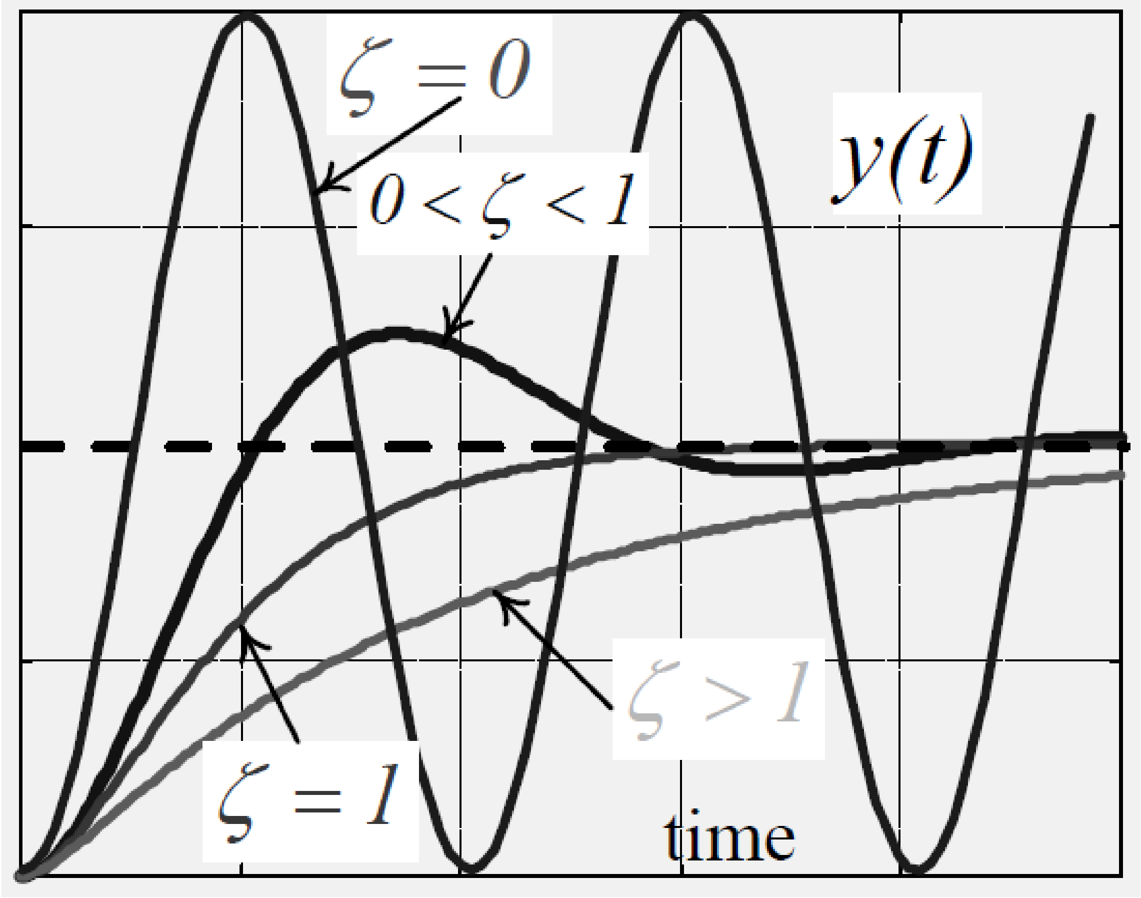
$$\phi = \cos^{-1}\zeta = 1.11\text{rad} = 63.4^o \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 2$$

$$y(t) = 1 - 1.118e^{-t}\sin(2t + 63.4^o)$$

Second Order System Step Responses

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

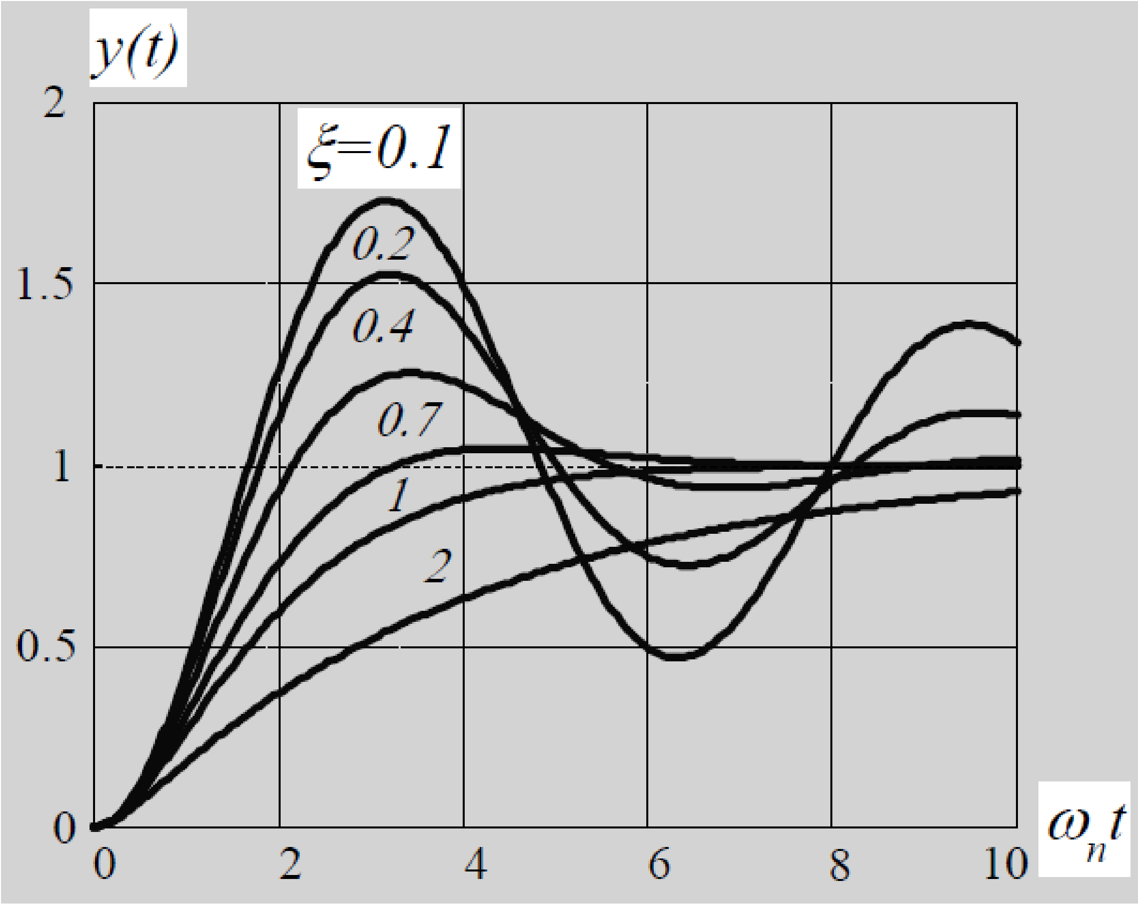


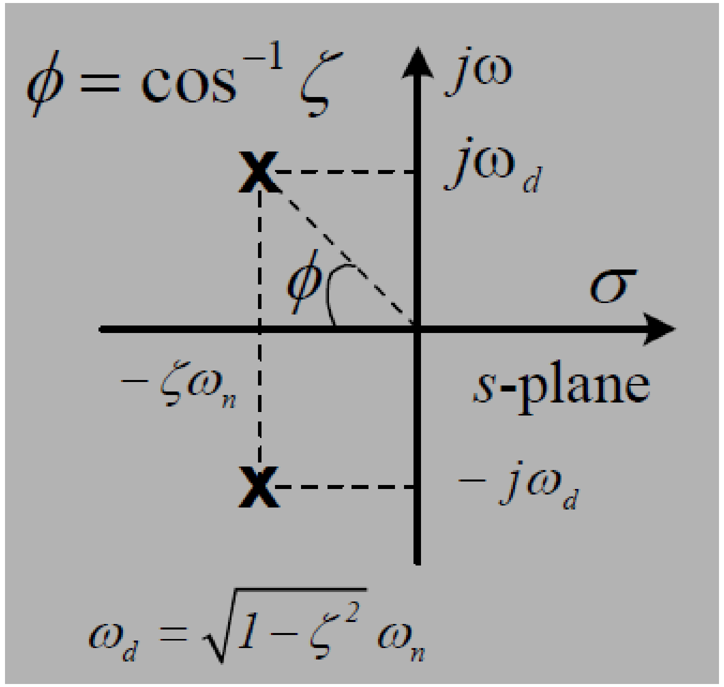


2nd order system pole position and response

- The unit step responses for various values of ζ:

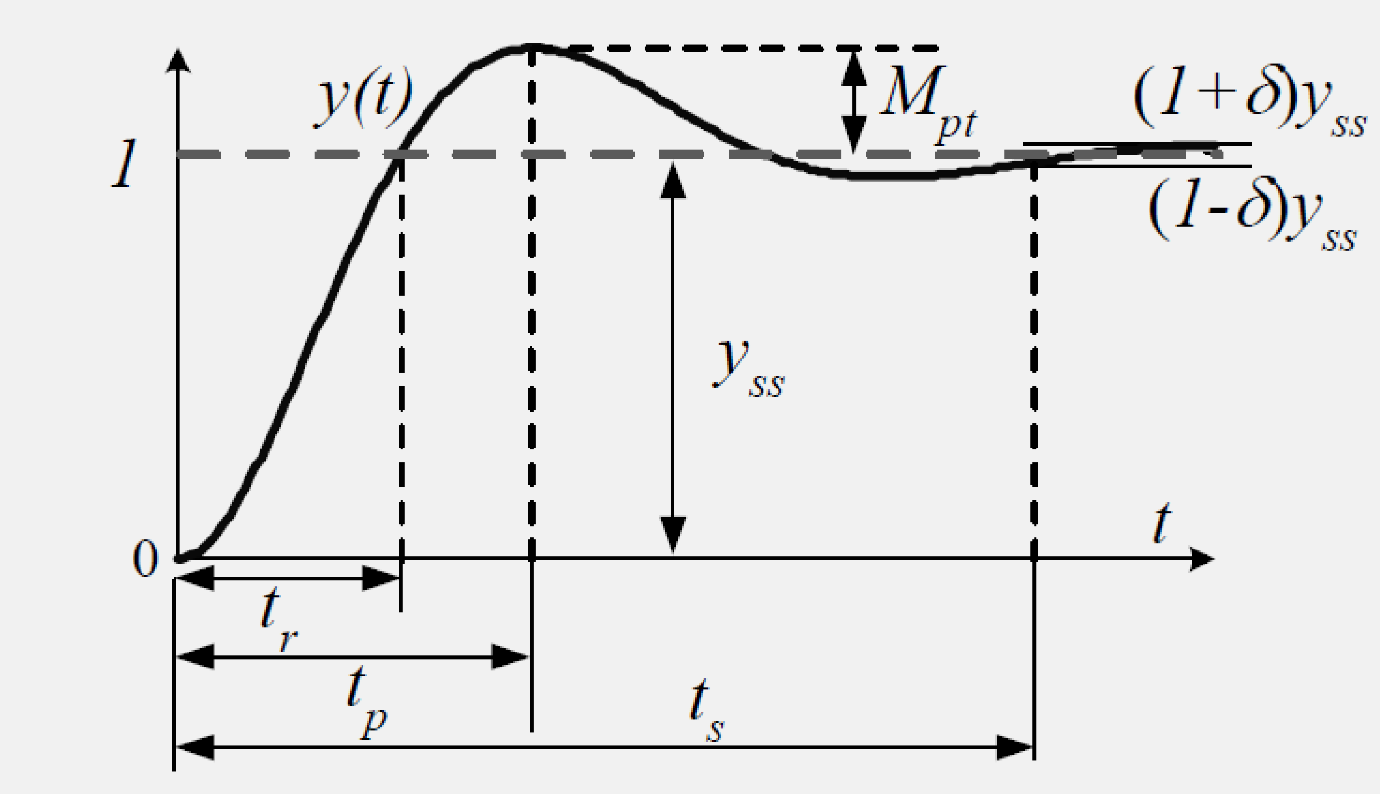
$$s_{1,2} = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n = -\zeta\omega_n \pm j\,\omega_d$$





Pole positions

2^{nd} order system unit step response characteri-stics



t_r	*rise time*
t_p	*time to first peak*
M_{pt}	*overshoot*
t_s	*settling time*

Four performance measures!

the response to fall within a prescribed band about the final steady value