Instrumentation and Controls

ETM 3301

Lecture 7

Instructor

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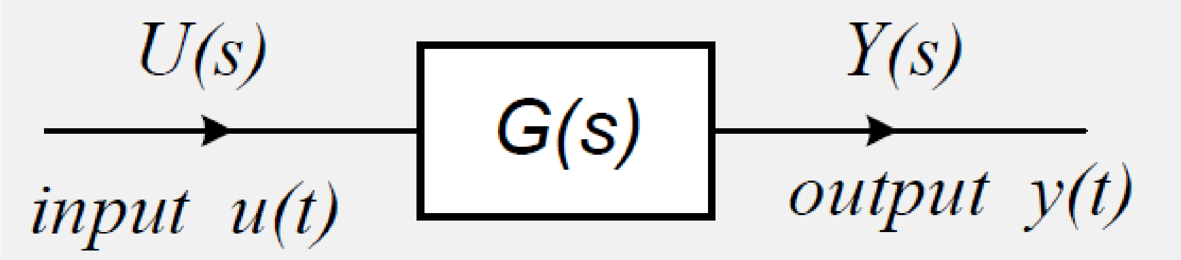
Chapter 3: System Time Responses, First and Second Order System Step Responses

- Time response concept
- First order system step response
- Standard second system transfer function, damping ratio and undamped natural frequency.
- Main characteristics of second order system step responses.
- Percentage overshoot and settling time.

Input-Output Method & Transfer Function (TF)

- The transfer function of a system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

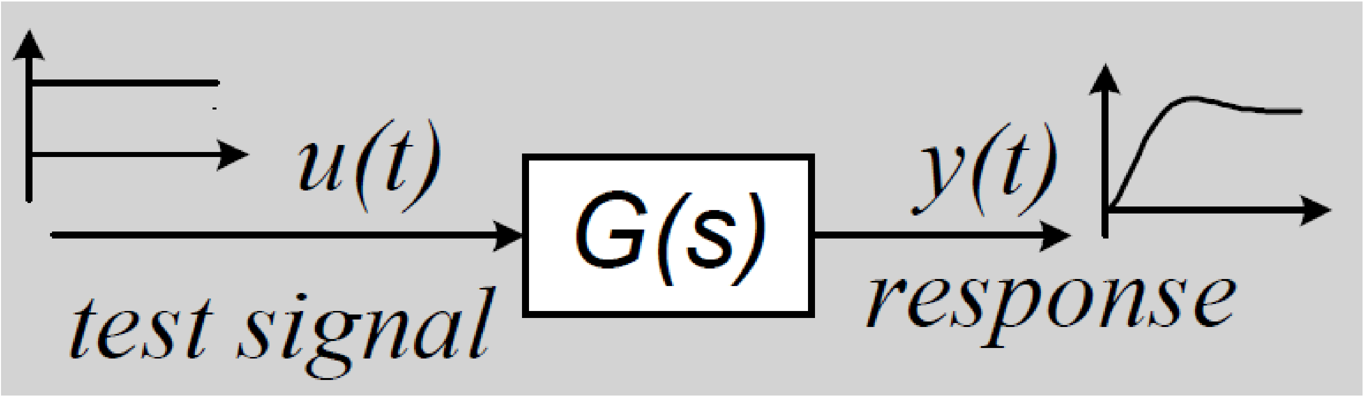


$$G(s) = \frac{Y(s)}{U(s)} = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}$$

Input	Output
Action	Consequence
Cause	Effect
Command	Response

Time Responses

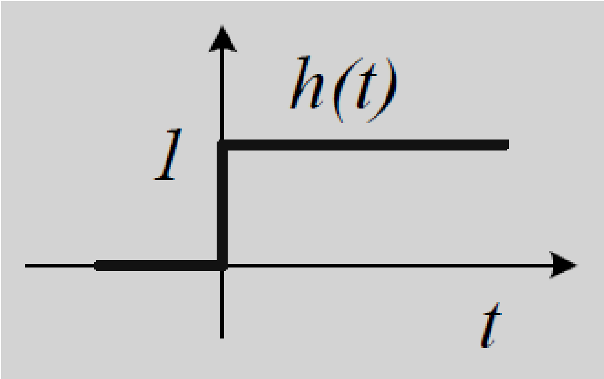
- It is important to know how a system responds to an input signal?
  - The output response is a function of time – time response.



- The system response is normally evaluated using standard test input signals.

Standard Test Signals (1)

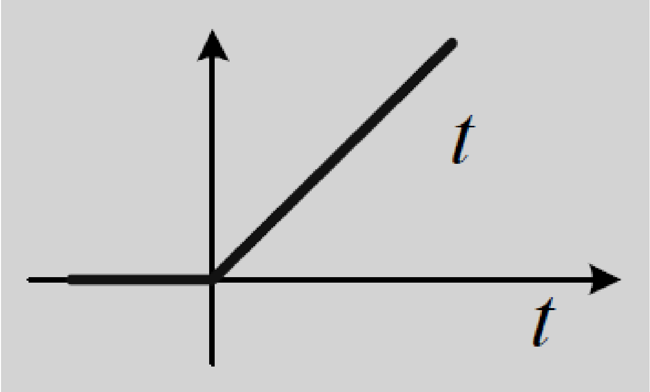
- **Step input:**



$$u(t) = h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

- **Rump input:**

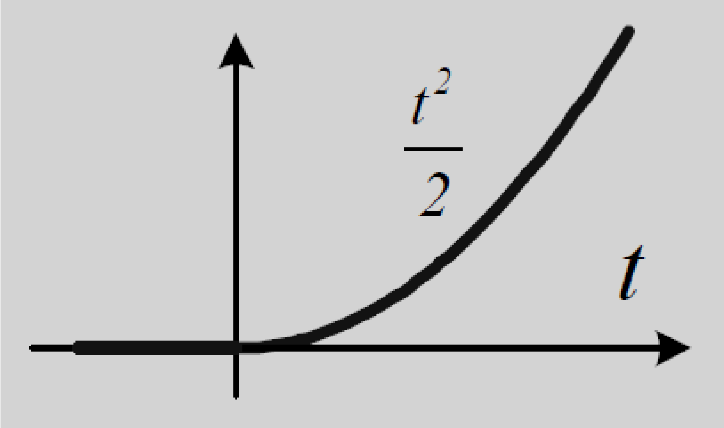


$$u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s^2}$$

Standard Test Signals (2)

- **Parabolic input :**



$$u(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

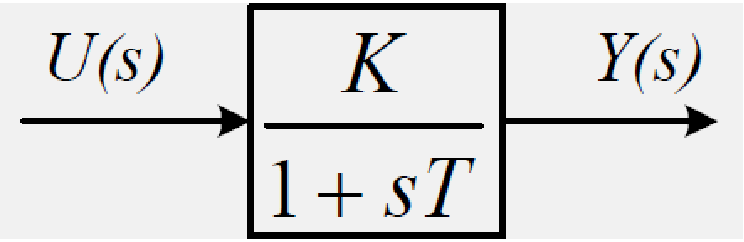
$$\mathcal{L}[u(t)] = \frac{1}{s^3}$$

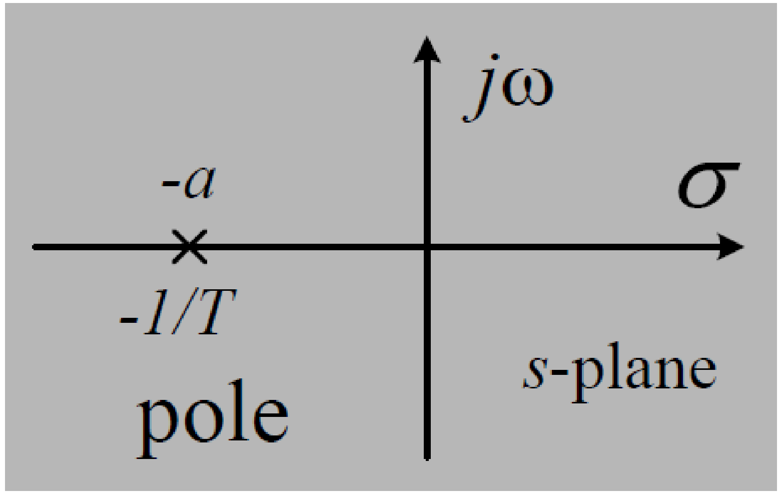
First order systems

- A first order system is described by the TF:

$$G(s) = \frac{K}{1+sT} \quad \text{Or} \quad G(s) = K\frac{a}{s+a} \qquad a = \frac{1}{T}$$

- We only consider the case when $T>0$, $a>0$.





Pole $-\frac{1}{T}$ or $-a$

- T: time constant; K: gain

First order system response to a step input, 1

- Unit step input $u(t) = 1$ $U(s) = \dfrac{1}{s}$

- Output response

$$Y(s) = G(s)U(s) = \frac{K}{s(1+sT)} = K\frac{1}{s(1+sT)}$$

$$= K\frac{(1+sT)-sT}{s(1+sT)} = K\left\{\frac{1}{s} - \frac{T}{1+sT}\right\}$$

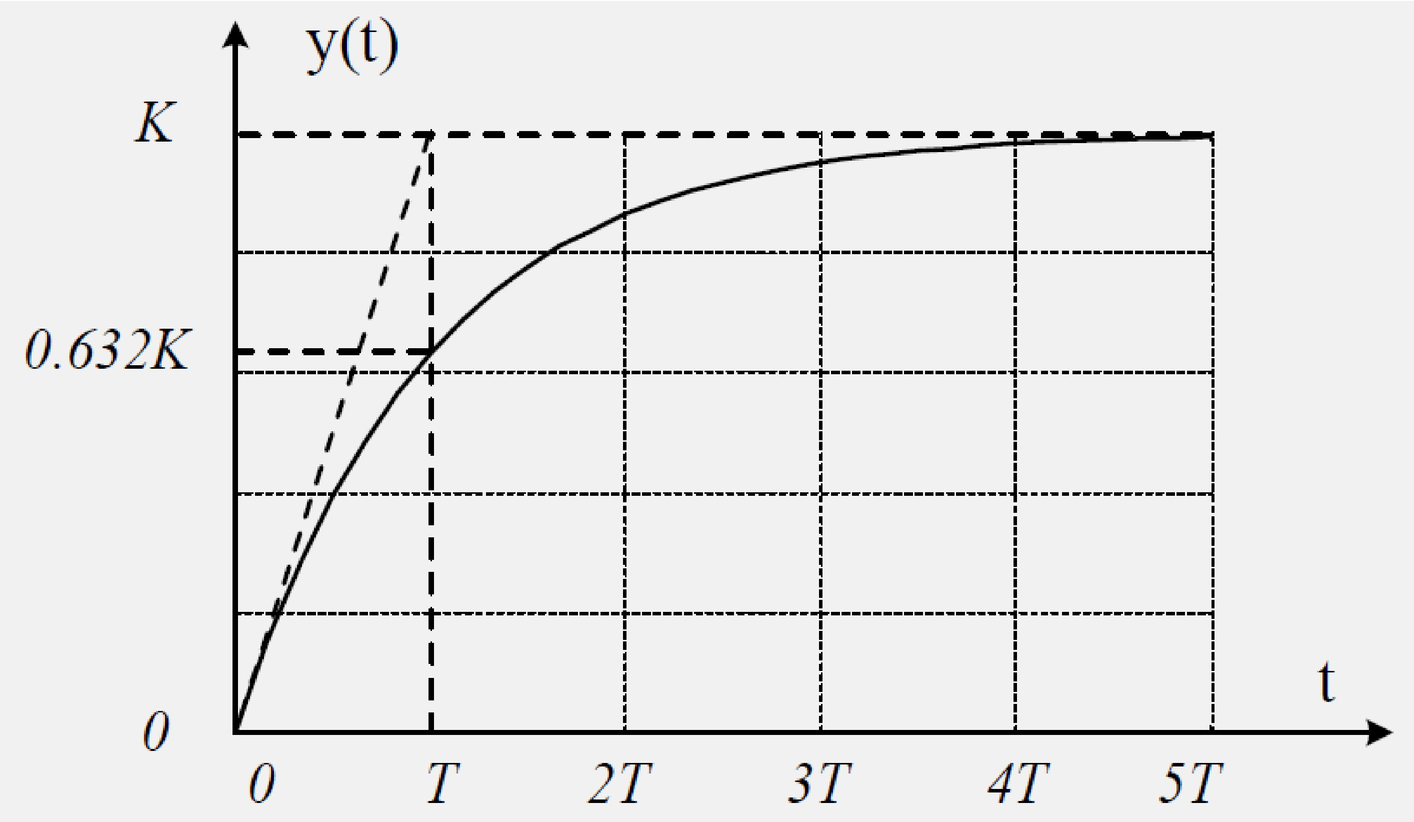
$$= K\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right\}$$

$$\frac{A-B}{C} = \frac{A}{C} - \frac{B}{C}$$

First order system response to a step input, 2

$$Y(s) = K\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = K\left(1 - e^{-\frac{t}{T}}\right)$$



First order system response to a step input, 3

- Output $y(t) = K\left(1 - e^{-\frac{t}{T}}\right)$
- Final (steady state) value $y_{ss} = \lim_{t \to \infty} y(t) = K$
- When $t = T$ $y(T) = K\left(1 - e^{-1}\right) = 0.632K = 0.632 y_{ss}$
  - the step response has reached to *63.2%* of its final value!
- T is defined as the **time constant**
  - the commonly used measure of the speed of the response.
  - The time constant is also given by the point where the tangent to the curve at $t=0$ (the initial gradient) meets the final value

First order system response to a step input, 4

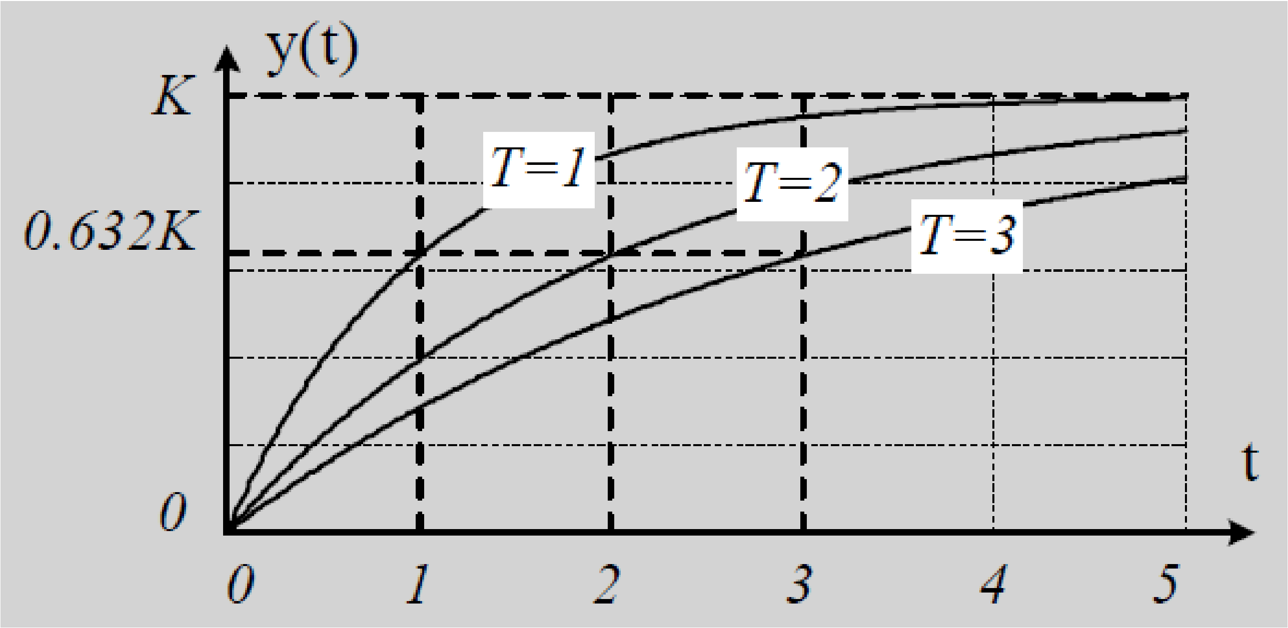
- Output $y(t) = K\left(1 - e^{-\frac{t}{T}}\right)$
- Final (steady state) value $y_{ss} = \lim_{t \to \infty} y(t) = K$
- When $t=4T$ $y(4T) = K\left(1 - e^{-4}\right) \approx 0.98K = 0.98y_{ss}$
- When $t=4.6T$ $y(4.6T) \approx 0.99y_{ss}$

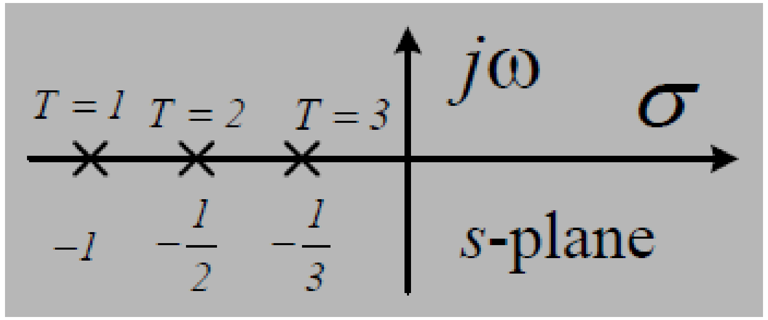
  – the step response reaches to *98%* of its final value with 4T.

  – the step response reaches to almost the final value with 4.6T.

First order system response to a step input, 5

- First order system $G(s) = \dfrac{K}{1 + sT}$
  - Write TF in this form means T can be seen directly.

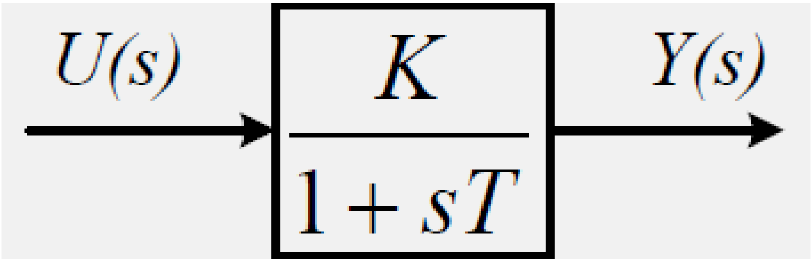




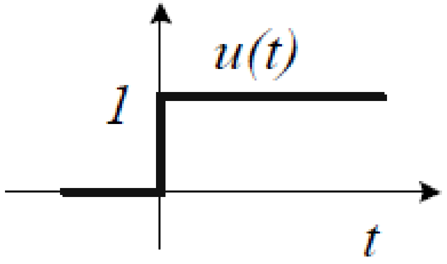
- The further to the left the system pole lies, the faster the response (the time constant is decreased).

First order system Step response

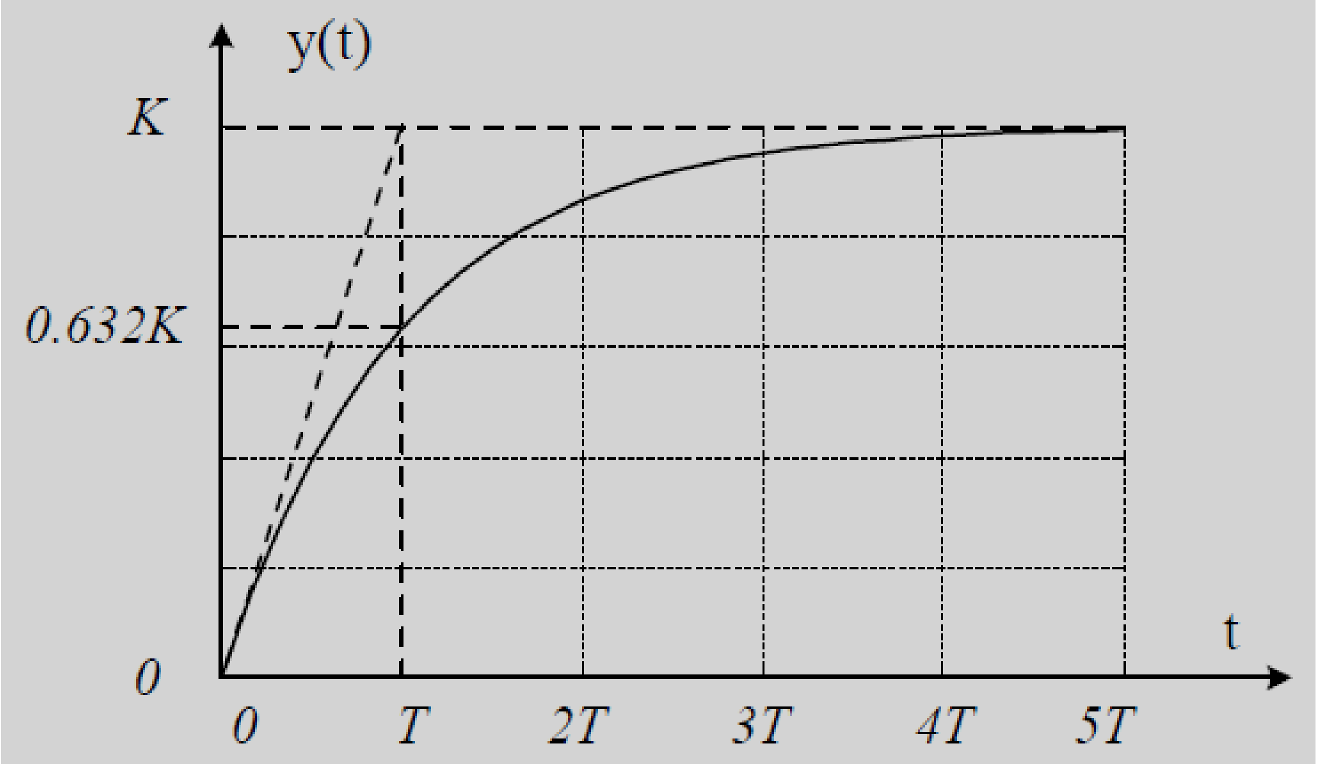
$$u(t) = 1$$



$$y(t) = K\left(1 - e^{-\frac{t}{T}}\right)$$



- Time constant T
- *Gain K*



Mass-Spring-Damper System Example

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + Bs + K}$$

Substituting parameters

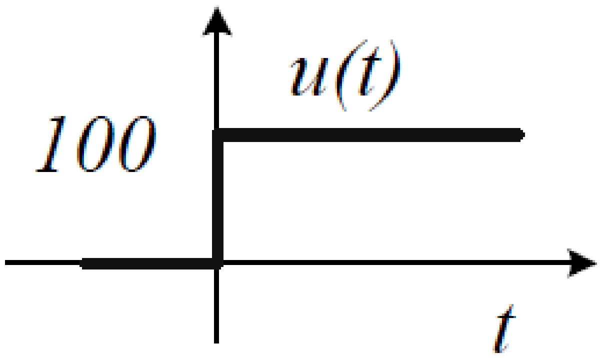
$M=800$ kg; $K=400$ N/m

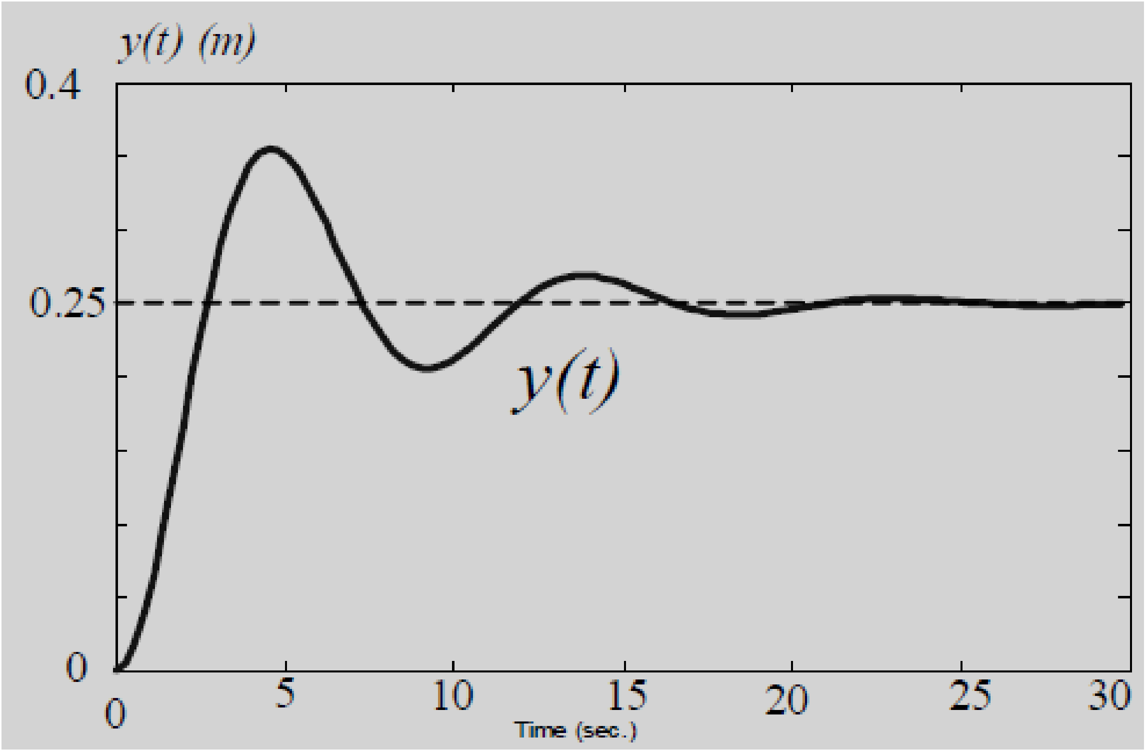
$B=300$ Ns/m

- Transfer function: $G(s) = \frac{0.00125}{s^2 + 0.0375s + 0.5}$
- Order =2; Second order system.
- Poles: $-0.01875 \pm 0.070686j$

Mass-Spring-Damper System Response

When *u=100N* (step input), system response:





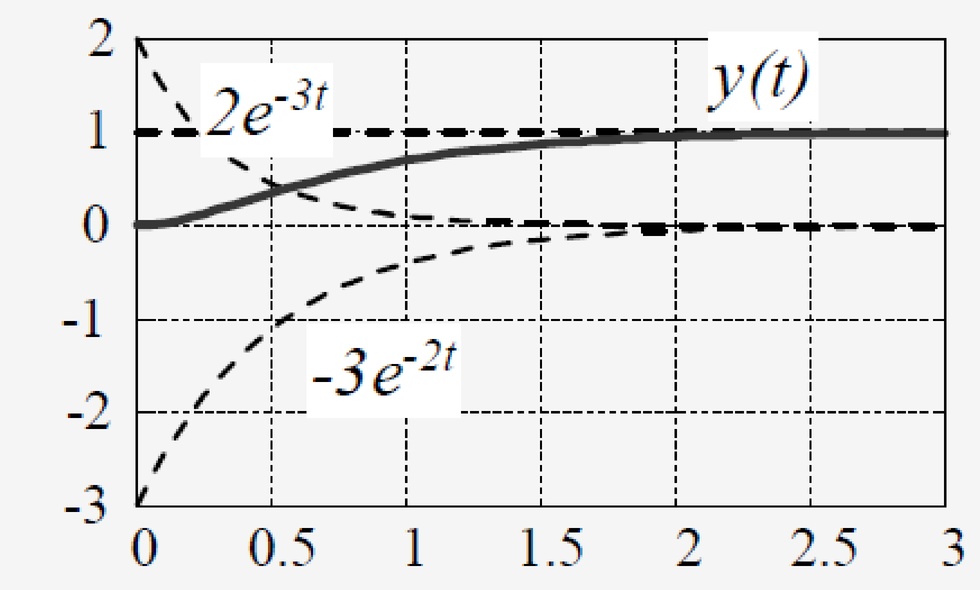
Second order system.

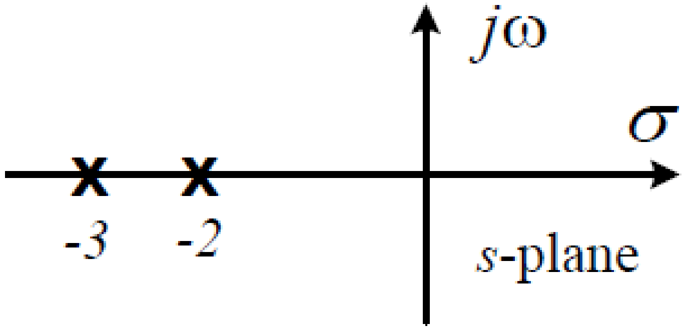
Second Order System Response Example 1-1

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^2 + 5s + 6} \qquad u(t) = 1; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{6}{s(s^2 + 5s + 6)} = \frac{6}{s(s+2)(s+3)} = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

$$y(t) = 1 - 3e^{-2t} + 2e^{-3t}$$

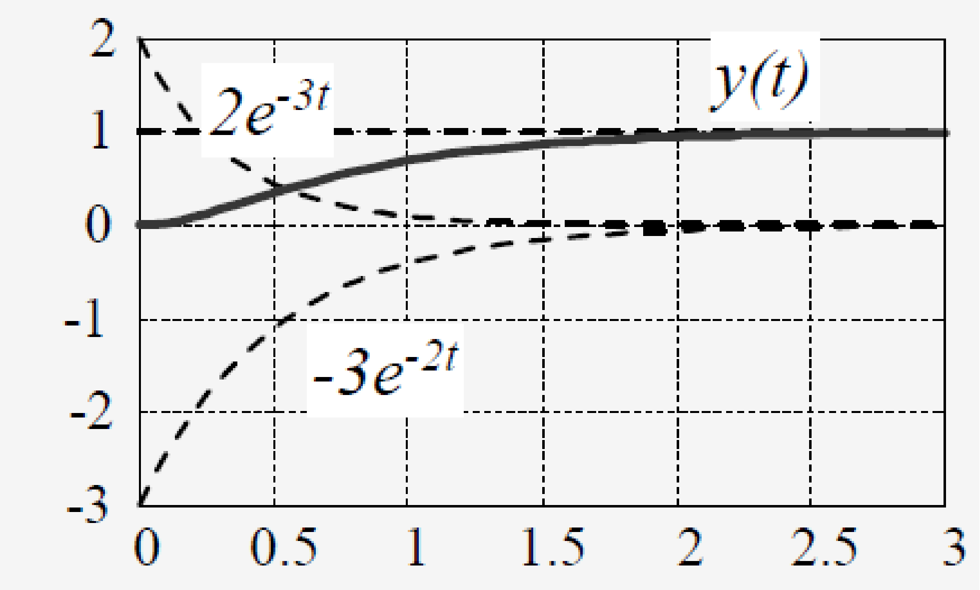




Pole positions

Second Order System Response Example 1-2

$$G(s) = \frac{6}{s^2 + 5s + 6} = \frac{6}{(s+2)(s+3)}$$



The response is similar to the first order system response

For the step response, this 2nd order system can be approximated by an 1st order system with a transfer function

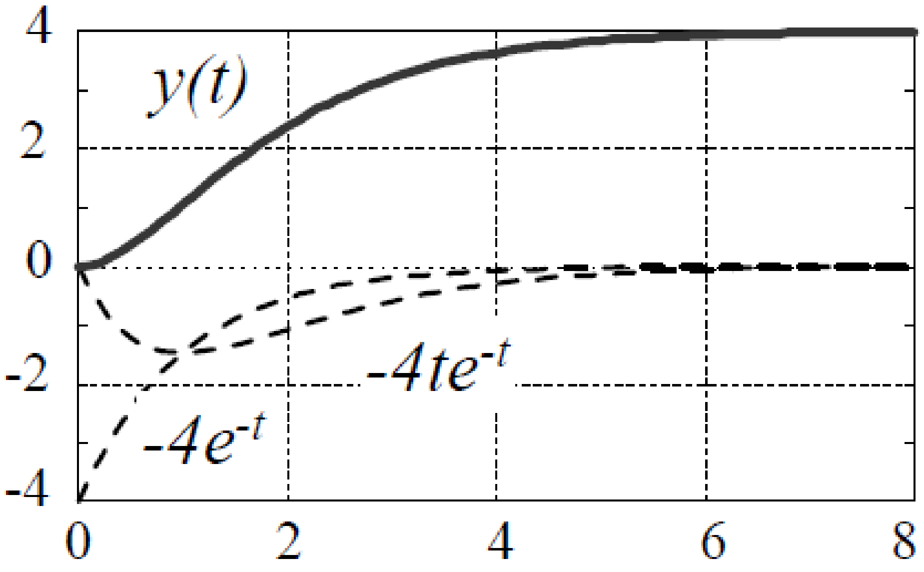
$$G(s) = \frac{1}{0.889s + 1}$$

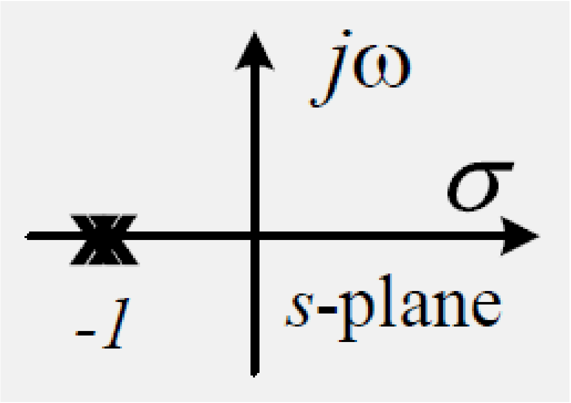
Second Order System Response Example 2

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4}{s^2 + 2s + 1} \qquad u(t) = 1; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{4}{s(s^2 + 2s + 1)} = \frac{4}{s(s+1)^2} = \frac{4}{s} - \frac{4}{s+1} - \frac{4}{(s+1)^2}$$

$$y(t) = 4(1 - e^{-t} - te^{-t})$$





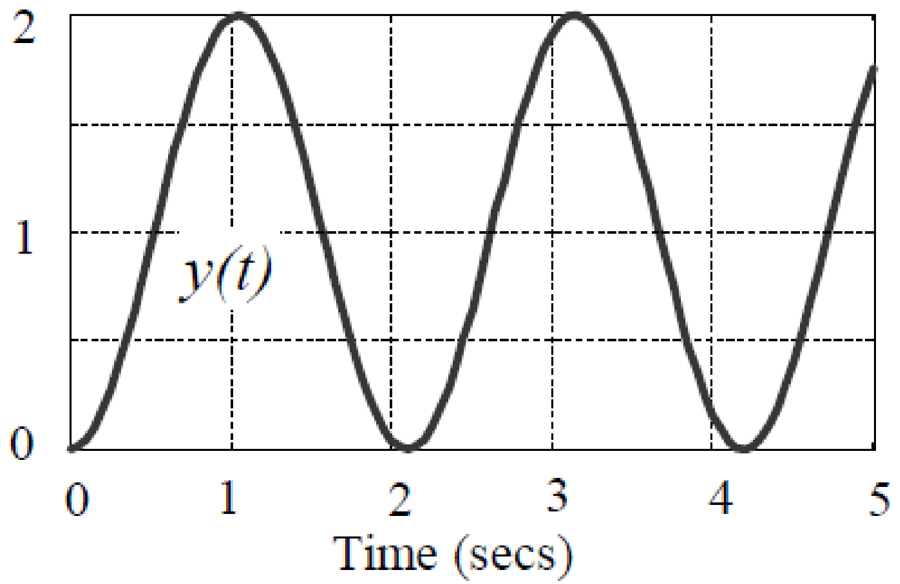
Pole positions

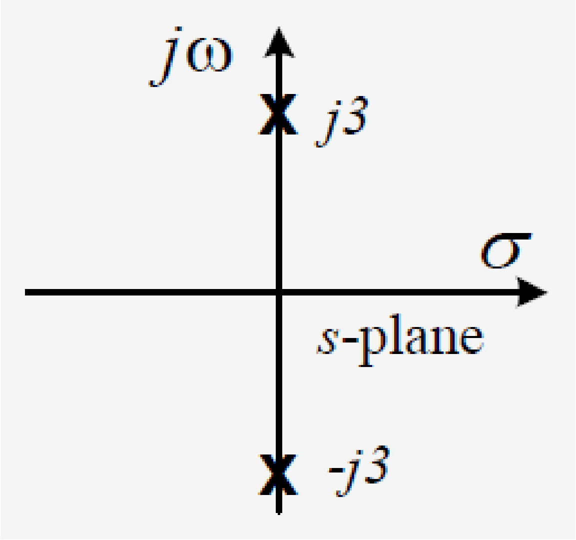
Second Order System Response Example 3

$$G(s) = \frac{Y(s)}{U(s)} = \frac{9}{s^2 + 9} \qquad u(t) = 1; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{9}{s(s^2+9)} = \frac{1}{s} - \frac{s}{s^2+9} = \frac{1}{s} - \frac{s}{s^2+3^2}$$

$$y(t) = 1 - \cos 3t$$





Pole positions

Second Order System Response Example 4(1)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5}{s^2 + 2s + 5} \qquad u(t) = 1 \, ; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{5}{s(s^2 + 2s + 5)} = \frac{s^2 + 2s + 5 - s^2 - 2s}{s(s^2 + 2s + 5)} = \frac{1}{s} - \frac{s + 2}{s^2 + 2s + 5}$$

$$= \frac{1}{s} - \frac{(s+1) + 1}{(s+1)^2 + (5-1)}$$

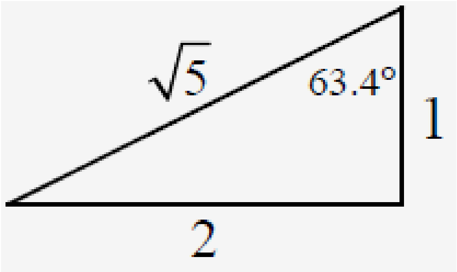
$$= \frac{1}{s} - \frac{(s+1)}{(s+1)^2 + 2^2} - \frac{1}{(s+1)^2 + 2^2}$$

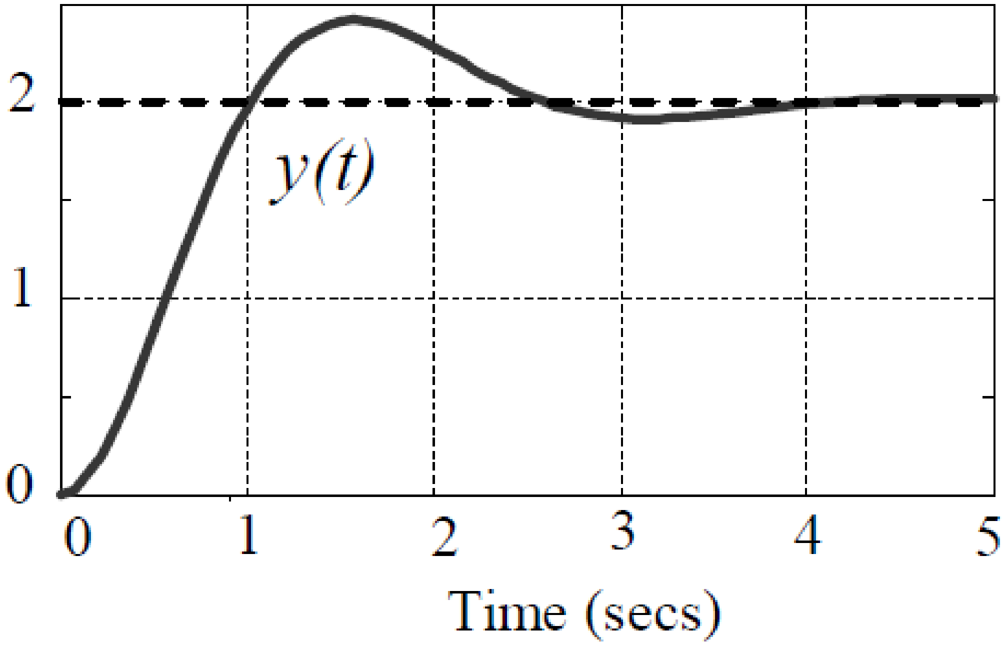
$$y(t) = 1 - e^{-t} \cos 2t - \frac{1}{2} e^{-t} \sin 2t$$

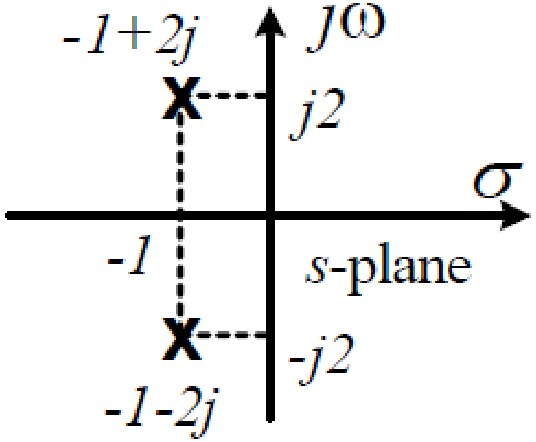
Second Order System Response Example 4(2)

$$y(t) = 1 - 1e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$\Rightarrow y(t) = 1 - \frac{\sqrt{5}}{2}e^{-t}\sin(2t + 63.4°)$$







Pole positions

Second Order System TF

- A 2nd order system transfer function (TF) is:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ *damping ratio*

ω_n *undamped natural frequency*