Instrumentation and Controls

ETM 3301

Lecture 5

Instructor

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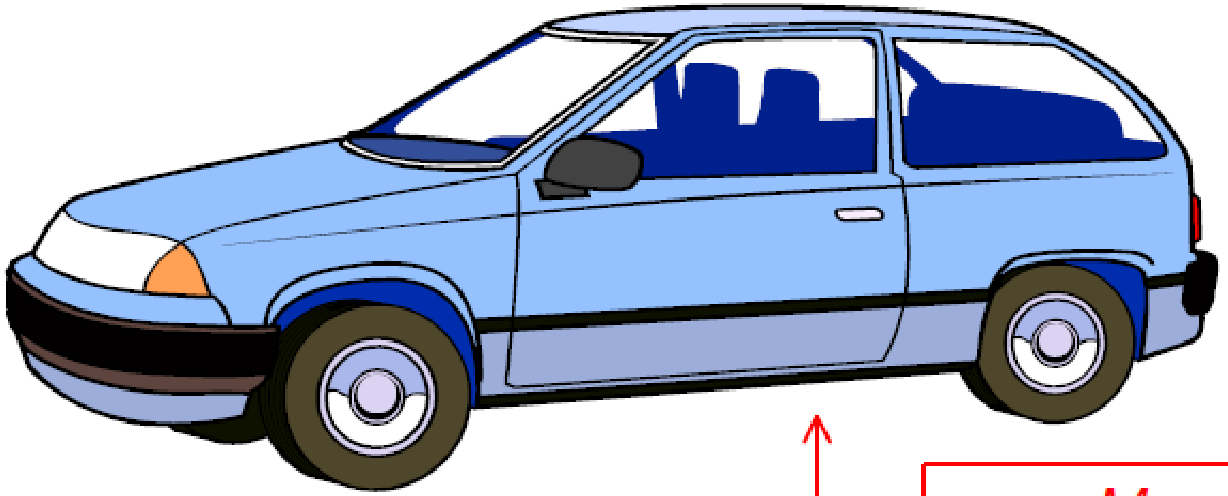
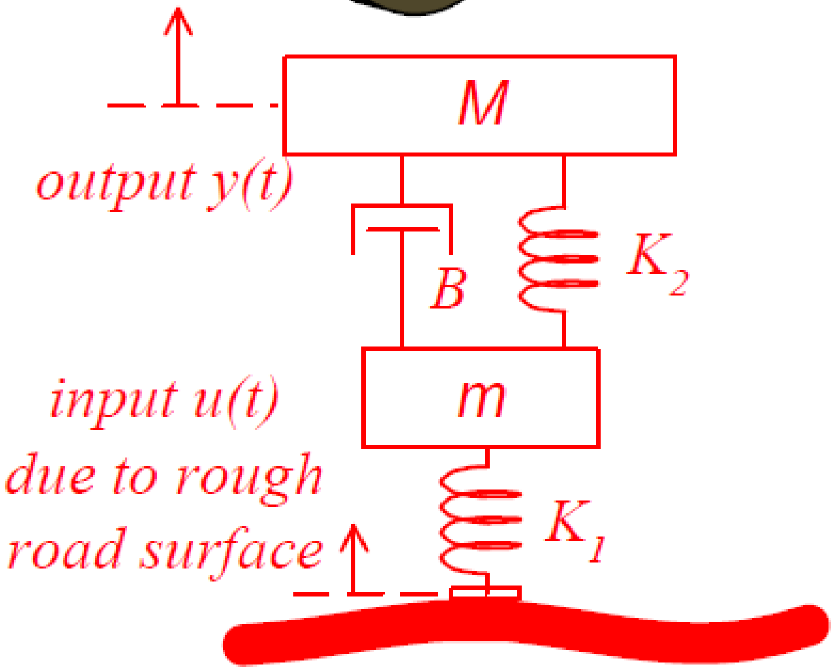
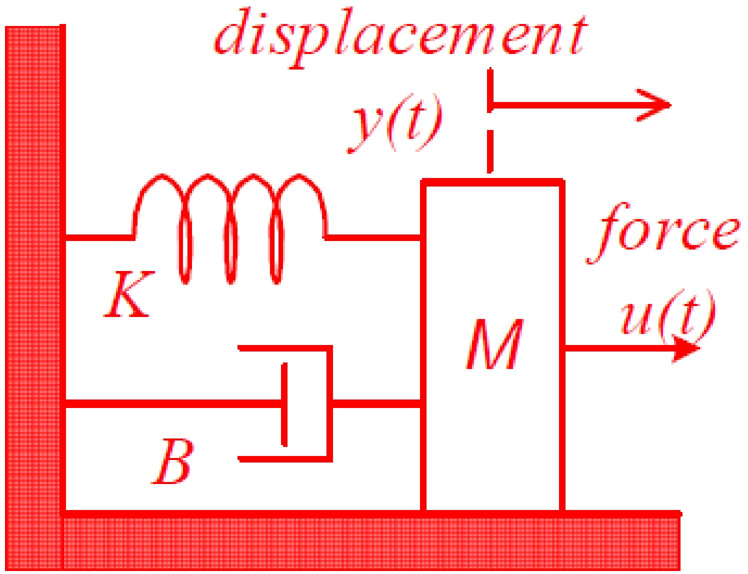
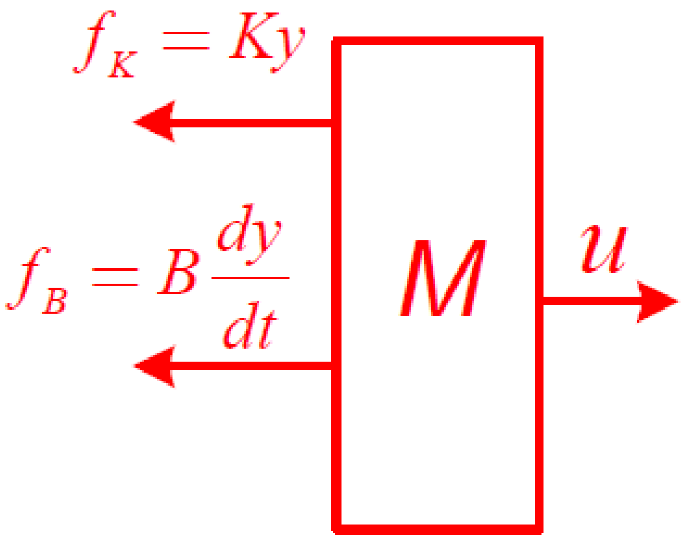
Car Suspension System Example





Car Suspension System Modelling
Simplified Case: *Mass-Spring-Damper System*





- To investigate the displacement *y(t)* caused by the force *u(t)*.

Newton's 2nd law: $M\frac{d^2y(t)}{dt^2} = u(t) - Ky(t) - B\frac{dy(t)}{dt}$

Mass-Spring-Damper System Modelling

$$M\frac{d^2y(t)}{dt^2} = u(t) - Ky(t) - B\frac{dy(t)}{dt}$$

Rearrange the equation:

$$M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{d(t)} + Ky(t) = u(t)$$

- Taking the Laplace transform of the both sides and assuming zero initial conditions.

$$Ms^2Y(s) + BsY(s) + KY(s) = U(s)$$

$$(Ms^2 + Bs + K)Y(s) = U(s)$$

$$Y(s) = \frac{1}{Ms^2 + Bs + K}U(s)$$

Mass-Spring-Damper System TF

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + Bs + K}$$

Substituting parameters

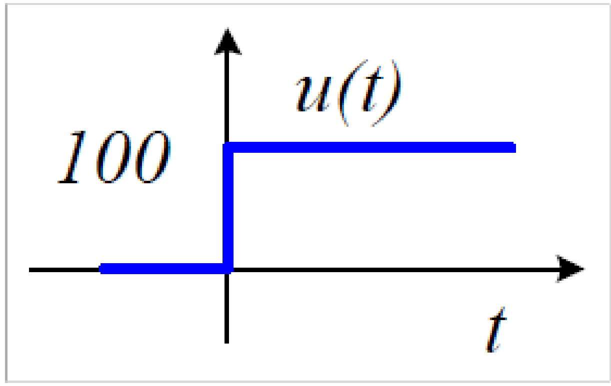
$M = 800$ kg; $K = 400$ N/m

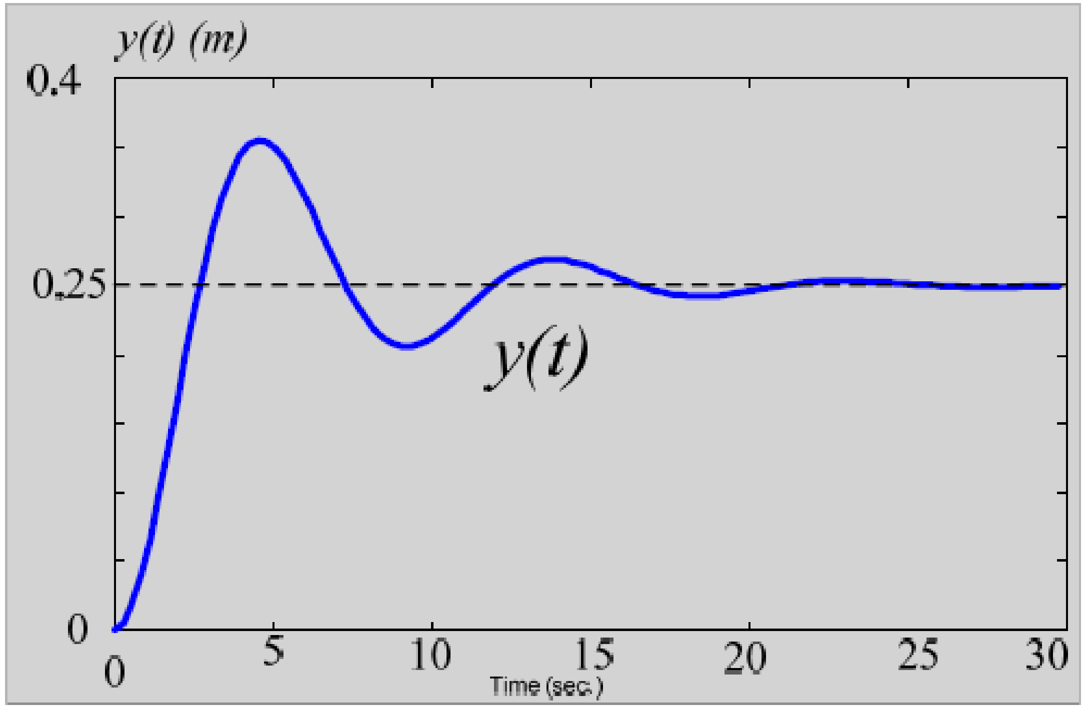
$B = 300$ Ns/m

- Transfer function: $$G(s) = \frac{0.125}{s^2 + 0.375s + 0.5}$$

Mass-Spring-Damper System Response

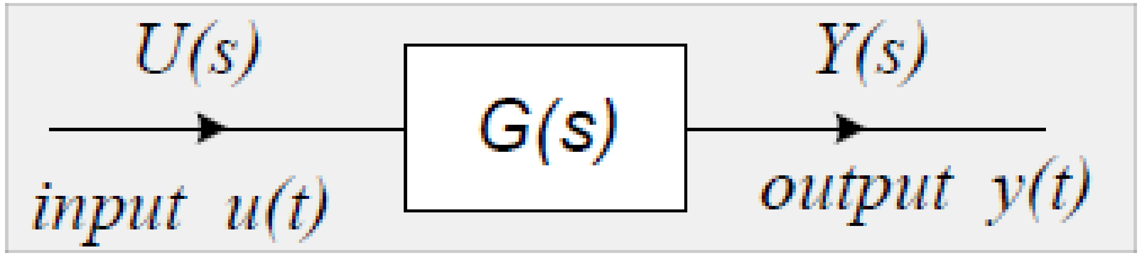
When $u = 100N$ (step input), system response:





Transfer Function (TF)

- The transfer function of a dynamic system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.



$$G(s) = \frac{Y(s)}{U(s)} = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}$$

Remarks on Transfer Function (TF)

- The transfer function relates the input to the output, however it does not provide any information concerning the physical structure of the system.
  - The transfer functions of many physically different systems can be identical.
- The transfer function is a property of a system itself, independent of the input.
- If the transfer function is known, the output of response can be found for different forms of input.

Derive Transfer Function (TF) (1)

- A *linear* dynamic system can be described by a linear differential equation (LDE).

$$\frac{d^3y(t)}{dt^3} + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_2\frac{d^2u(t)}{dt^2} + b_1\frac{du(t)}{dt} + b_0u(t)$$

- Assume all initial conditions are zero.

$$\frac{d^2y(0)}{dt^2} = \frac{dy(0)}{dt} = y(0) = 0; \quad \frac{du(0)}{dt} = u(0) = 0$$

Derive Transfer Function (TF) (2)

- Define: $\mathcal{L}[y(t)] = Y(s); \quad \mathcal{L}[u(t)] = U(s)$

$$\mathcal{L}\left\{\frac{d^3y(t)}{dt^3}\right\} = s^3Y(s); \quad \mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s)$$

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s)$$

$$\mathcal{L}\left\{\frac{d^2u(t)}{dt^2}\right\} = s^2U(s); \quad \mathcal{L}\left\{\frac{du(t)}{dt}\right\} = sU(s)$$

Derive Transfer Function (TF) (3)

$$\frac{d^3y(t)}{dt^3} + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t)$$
$$= b_2\frac{d^2u(t)}{dt^2} + b_1\frac{du(t)}{dt} + b_0u(t)$$

Applying Laplace Transform of both sides of LDE.

$$s^3Y(s) + a_2s^2Y(s) + a_1sY(s) + a_0Y(s)$$
$$= b_2s^2U(s) + b_1sU(s) + b_0U(s)$$

- The Laplace Transform converts a LDE into an algebraic equation of s.

Derive Transfer Function (TF) (4)

$$s^3Y(s)+a_2s^2Y(s)+a_1sY(s)+a_0Y(s)$$
$$=b_2s^2U(s)+b_1sU(s)+b_0U(s)$$

$$(s^3+a_2s^2+a_1s+a_0)Y(s)=(b_2s^2+b_1s+b_0)U(s)$$

$$Y(s)=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}U(s)$$

- Finally, the transfer function is

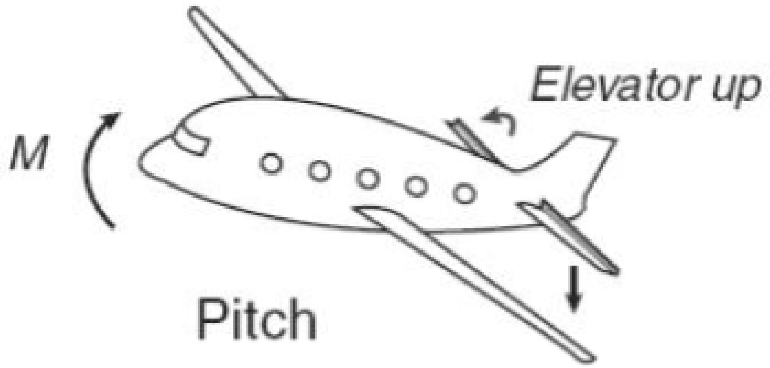
$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$$

Aircraft Pitch Angle/Rate Control Example

Pitch rate $q(t)$

Elevator angle $\eta(t)$

Vertical speed $w(t)$



$$\dot{w}(t) = -0.8w(t) + 238q(t) - 24\eta(t) \qquad (eq1)$$

$$\dot{q}(t) = -0.07w(t) - 9.2q(t) - 4\eta(t) \qquad (eq2)$$

Apply Laplace transform to (eq1)

$$sW(s) = -0.8W(s) + 238Q(s) - 24\eta(s)$$

$$W(s) = \frac{238}{s+0.8}Q(s) - \frac{24}{s+0.8}\eta(s) \qquad (eq3)$$

Aircraft Pitch Angle/Rate Control Example

Apply Laplace transform to (eq2)

$$sQ(s) = -0.07W(s) - 9.2Q(s) - 4\eta(s) \qquad (eq4)$$

Substitute (eq3) into (eq4)

$$sQ(s) = -0.07\left(\frac{238}{s+0.8}Q(s) - \frac{24}{s+0.8}\eta(s)\right) - 9.2Q(s) - 4\eta(s)$$

$$sQ(s) = -\frac{16.66}{s+0.8}Q(s) + \frac{1.68}{s+0.8}\eta(s) - 9.2Q(s) - 4\eta(s)$$

$$s(s+1.8)Q(s) = -16.66Q(s) + 1.68\eta(s)$$
$$- 9.2(s+0.8)Q(s) - 4(s+0.8)\eta(s)$$

Aircraft Pitch Angle/Rate Control Example

Apply Laplace transform to (eq2)

$$sQ(s) = -0.07W(s) - 9.2Q(s) - 4\eta(s) \qquad (eq4)$$

Substitute (eq3) into (eq4)

$$sQ(s) = -0.07\left(\frac{238}{s+0.8}Q(s) - \frac{24}{s+0.8}\eta(s)\right) - 9.2Q(s) - 4\eta(s)$$

$$sQ(s) = -\frac{16.66}{s+0.8}Q(s) + \frac{1.68}{s+0.8}\eta(s) - 9.2Q(s) - 4\eta(s)$$

$$s(s+0.8)Q(s) = -16.66Q(s) + 1.68\eta(s)$$
$$-9.2(s+0.8)Q(s) - 4(s+0.8)\eta(s)$$

Aircraft Pitch Angle/Rate Control Example

$$s(s+1.8)Q(s) = -16.66Q(s) + 1.68\eta(s)$$
$$-9.2(s+0.8)Q(s) - 4(s+0.8)\eta(s)$$

$$(s^2 + 0.8s + 16.66 + 9.2s + 9.2\times 0.8)Q(s)$$
$$= (1.68 - 4s - 3.2)\eta(s)$$

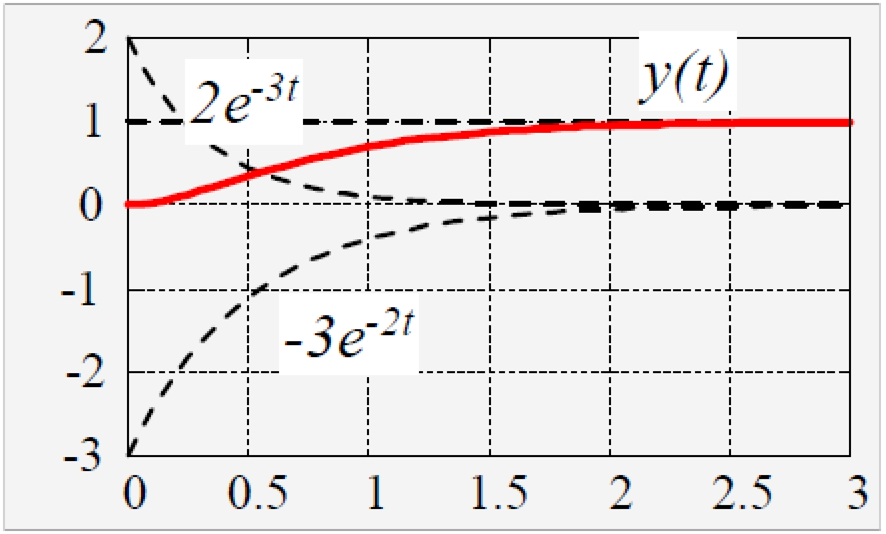
$$Q(s) = \frac{-4s - 1.57}{s^2 + 10s + 24}\eta(s)$$

Responses and Transfer function

System 1 $G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^2 + 5s + 6}$ $u(t) = 1\,; \quad U(s) = \frac{1}{s}$

$$Y(s) = \frac{6}{s(s^2 + 5s + 6)} = \frac{6}{s(s+2)(s+3)} = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

$$y(t) = 1 - 3e^{-2t} + 2e^{-3t}$$



Responses and Transfer function

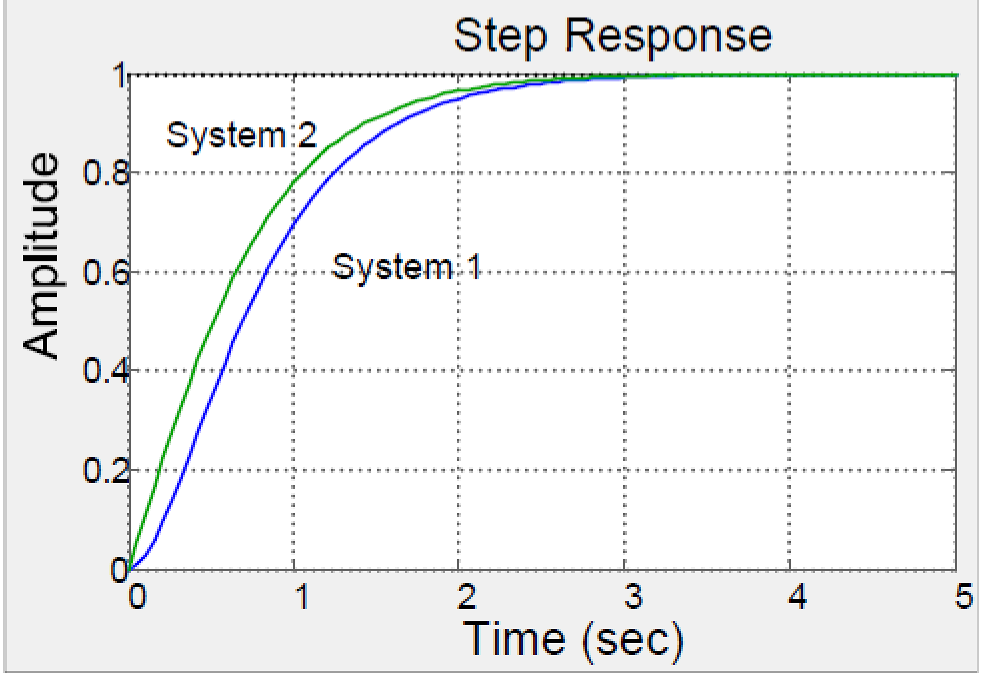
System 2

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+6}{s^2+5s+6}$$

$$u(t) = 1\,; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{s+6}{s(s^2+5s+6)} = \frac{s+6}{s(s+2)(s+3)} = \frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+3}$$

$$y(t) = 1 - 2e^{-2t} + e^{-3t}$$



The system response characteristics mainly determined by the transfer function denominator.

Responses and Transfer function

System 3

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5}{s^2 + 2s + 5}$$

$$u(t) = 1\,; \quad U(s) = \frac{1}{s}$$

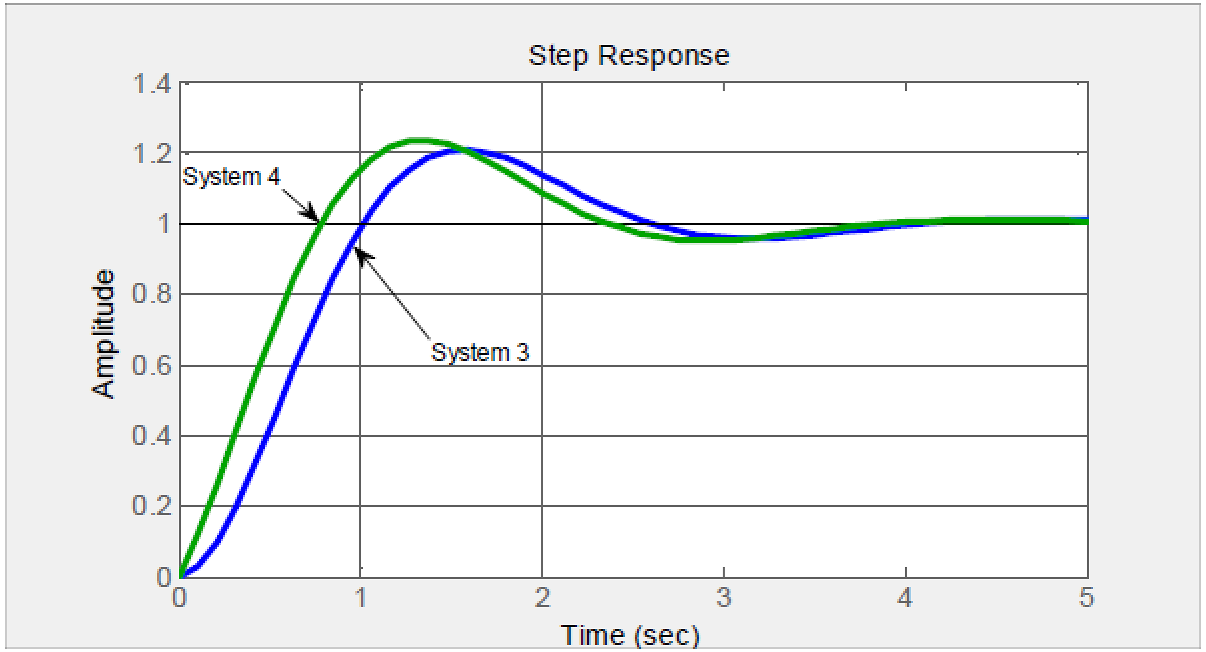
$$Y(s) = \frac{5}{s(s^2 + 2s + 5)} = \frac{1}{s} - \frac{s+1}{(s+1)^2 + 2^2} - \frac{1}{(s+1)^2 + 2^2}$$

$$y(t) = 1 - 1.18e^{-t}\sin(2t + 63.4°)$$

System 4

$$G(s) = \frac{s+5}{s^2 + 2s + 5}$$

The system response characteristics mainly determined by the transfer function denominator.



Responses and Transfer function

System 5

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3}{s^2 + 4s + 3}$$

$$u(t) = 1\,; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{3}{s(s+1)(s+3)} = \frac{1}{s} - \frac{3}{2}\frac{1}{s+1} + \frac{1}{2}\frac{2}{s+3}$$

$$y(t) = 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

System 6

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s+3}{s^2 + 4s + 3}$$

$$u(t) = 1\,; \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{2s+3}{s(s+1)(s+3)} = \frac{1}{s} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}$$

$$y(t) = 1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

The system response characteristics mainly determined by the transfer function denominator.