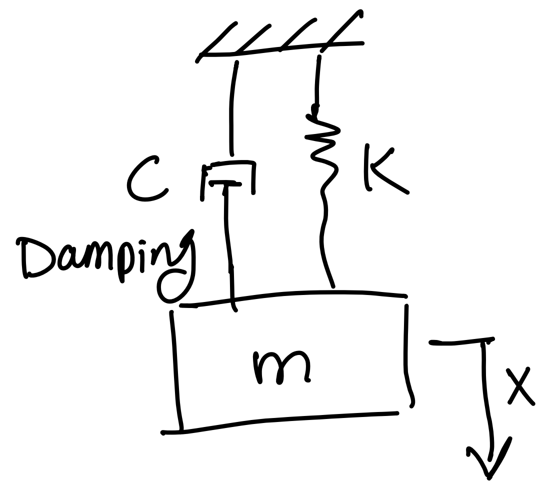
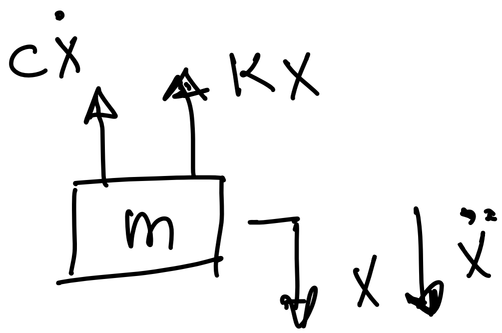
Damped Free vibration:

Damping Force $F_d = c\dot{x}$



F.B.D



$$\stackrel{+}{\uparrow} \sum F_x = m\ddot{x} \implies -c\dot{x} - Kx = m\ddot{x}$$

Rearrange: $m\ddot{x} + c\dot{x} + Kx = 0$

No external Force

$F(t) = 0$

Solution:

If $x = e^{st}$ is the solution

what is $s = ?$

$$x = e^{st} \rightarrow \dot{x} = se^{st} \longrightarrow \ddot{x} = s^2 e^{st}$$

$$\underbrace{(ms^2 + cs + k)}_{=0} \underbrace{e^{st}}_{\neq 0} = 0$$

$$ms^2 + cs + K = 0$$

or

$$s^2 + \frac{c}{m}s + \frac{K}{m} = 0$$

$$\Rightarrow \quad s_{1,2} = -\frac{c}{2m} \mp \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}$$

The general solution:

$$X = \underbrace{A}_{?} e^{s_1 t} + \underbrace{B}_{?} e^{s_2 t}$$

Initial conditions: $\begin{cases} \text{Initial Displacement} \\ X(t=0) = X_0 \\ \text{Initial velocity} \\ \dot{X}(t=0) = \dot{X}_0 \end{cases}$

$$X(0) = A + B = X_0$$

$$\dot{X} = A s_1 e^{s_1 t} + B s_2 e^{s_2 t} = A s_1 + B s_2 = \dot{X}_0$$

A and B are constants to be evaluated from the initial conditions $x(0)$ and $\dot{x}(0)$

From above we have:

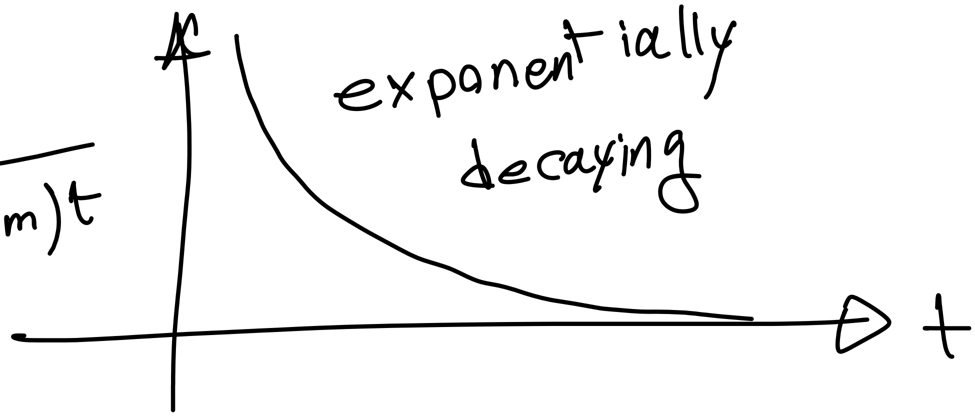
$$x = Ae^{s_1 t} + Be^{s_2 t}$$

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

substitute s_1 and s_2 in x:

$$x = e^{-(c/2m)t}\left(Ae^{\left(\sqrt{(c/2m)^2 - k/m}\right)t} + Be^{-\left(\sqrt{(c/2m)^2 - k/m}\right)t}\right)$$

$$e^{-(c/2m)t} = \frac{1}{e^{(c/2m)t}}$$



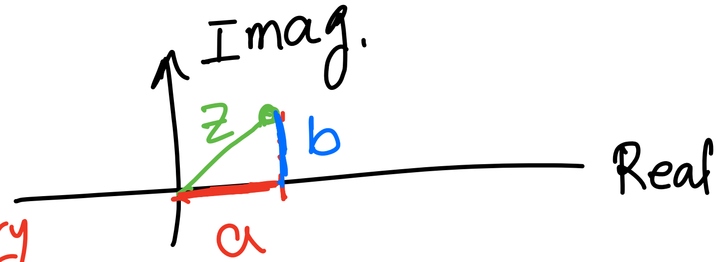
The response also depends on if the numerial value within the radical is positive, zero, or negative.

$\sqrt{-1} = i$ imaginary

The general form of a complex number

$Z = a + ib$

Real Imaginary



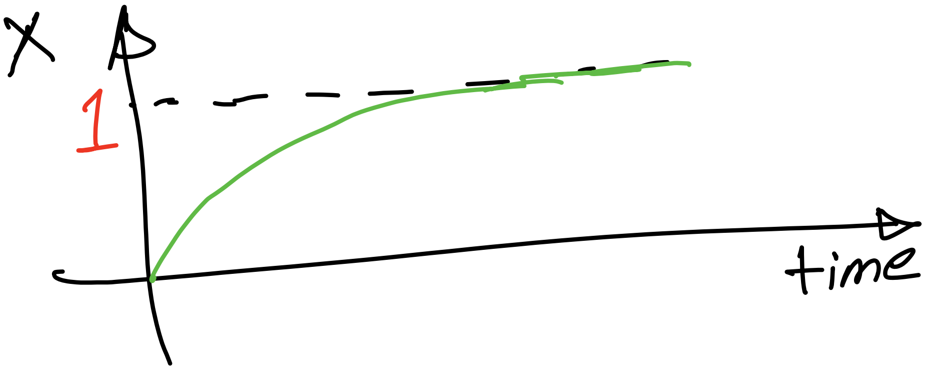
$\sqrt{(c/2m)^2 - K/m}$

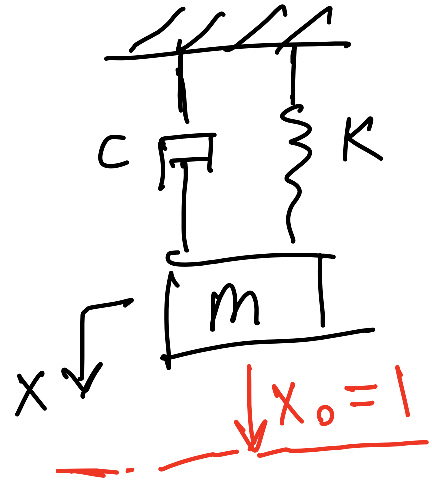
when $\left(\frac{c}{2m}\right)^2 > \frac{K}{m}$

$(c/2m)^2 - K/m > 0 \Rightarrow$

$\sqrt{(c/2m)^2 - K/m} \rightarrow$ Real

If Real there will be No oscilation.





we reffer to this case as overdamped.

when $\left(\frac{c}{2m}\right)^2 < \frac{K}{m}$

the exponent becams an imaginary

$$\sqrt{(c/2m)^2 - K/m}$$

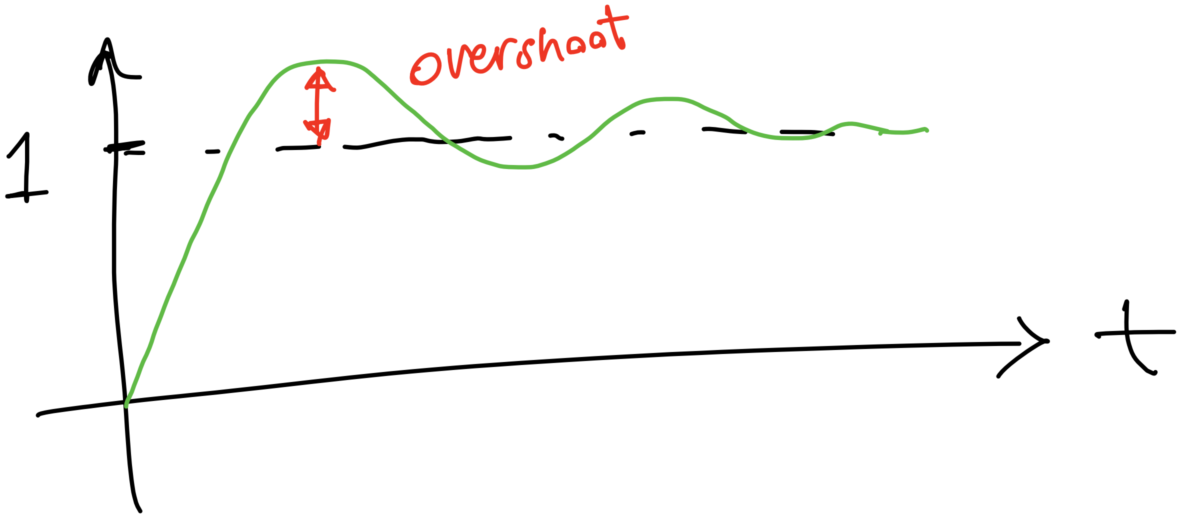
$$\boxed{i = \sqrt{-1}}$$

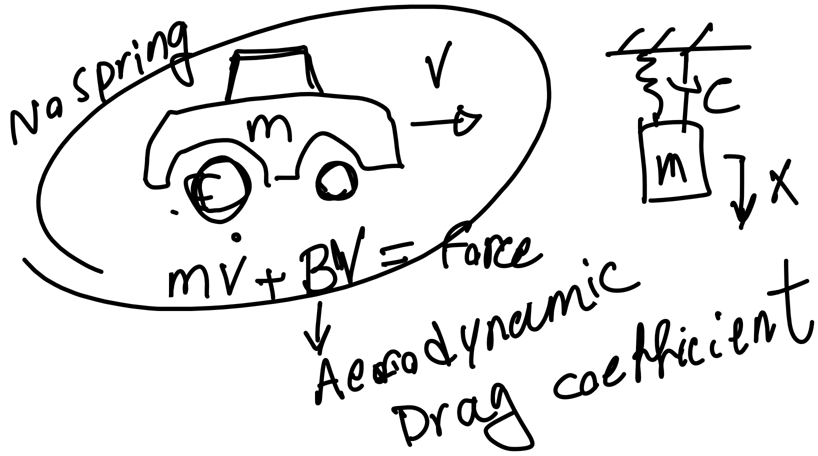
$$\left(\frac{c}{2m}\right)^2 - K/m < 0$$

$$\sqrt{\left(\frac{c}{2m}\right)^2 - K/m} \longrightarrow i\sqrt{\frac{K}{m} - \left(\frac{c}{2m}\right)^2}$$

$$e^{\pm i\left(\sqrt{\frac{K}{m} - \left(\frac{c}{2m}\right)^2}\right)t}$$

we refer to this case underdamped





In the limiting case between nonoscilation and oscilation :

$$\left(\frac{C_c}{2m}\right)^2 = \frac{K}{m}$$

$$\frac{C_c}{2m} = \underbrace{\sqrt{\frac{K}{m}}}_{\omega_n} \quad \text{or} \quad C_c = 2m\,\omega_n$$

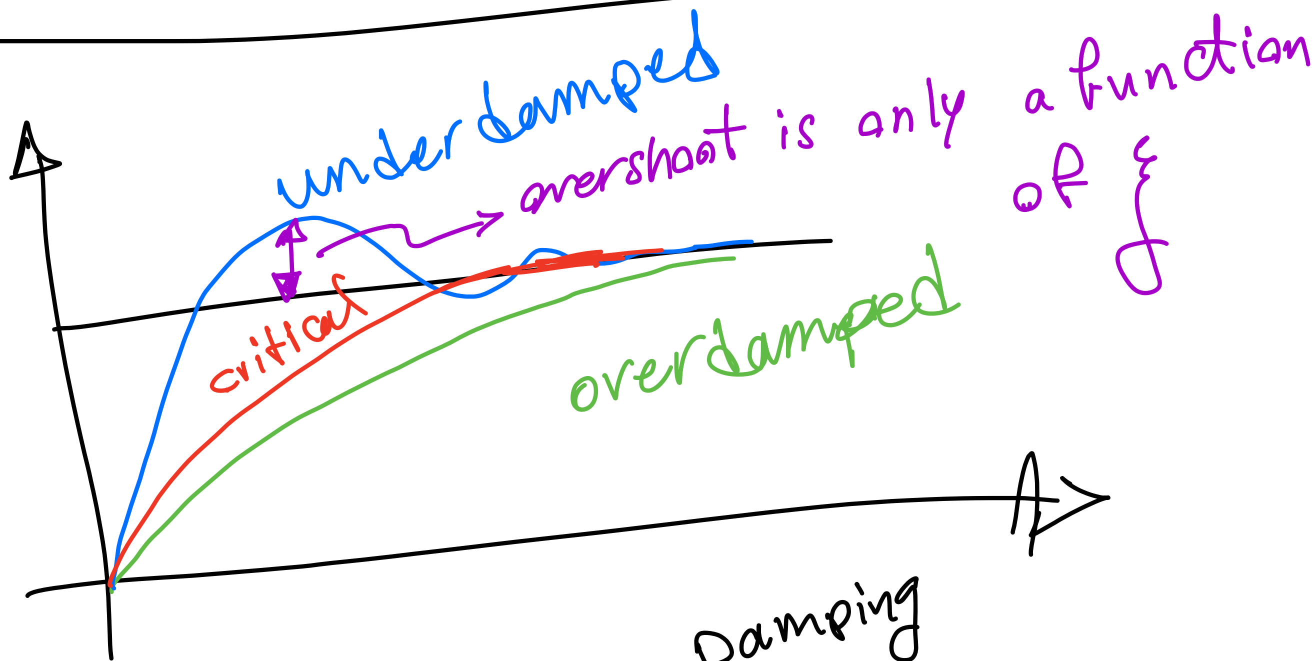
Natural frequency

or

$$C_c = 2\sqrt{Km}$$

critical Damping

$$2m\sqrt{\frac{K}{m}} = 2\sqrt{\frac{Km^2}{m}} = 2\sqrt{Km}$$



$$\xi = \frac{C}{C_c}$$

ξ → Damping ratio

C → Damping

C_c → critial damping

$$\frac{C}{2m} = \xi\left(\frac{C_c}{2m}\right) = \xi\,\omega_n$$

$$\boxed{\xi = \frac{C}{C_c} \Rightarrow \xi C_c = C \Rightarrow \frac{C}{2m} = \xi\left(\frac{C_c}{2m}\right)}$$

$$C_c = 2\sqrt{Km}$$

$$\frac{C}{2m} = \zeta\left(\frac{2\sqrt{Km}}{2m}\right) = \zeta\,\frac{\sqrt{Km}}{\sqrt{m^2}} = \zeta\sqrt{\frac{Km}{m^2}}$$

$$\frac{C}{2m} = \zeta\sqrt{\frac{K}{m}} \qquad \sqrt{\frac{K}{m}} = \omega_n$$

$$\boxed{\frac{C}{2m} = \zeta\,\omega_n}$$

Differential equation of motion

$$m\ddot{x} + c\dot{x} + Kx = 0$$

$$\rightarrow \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{K}{m}x = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

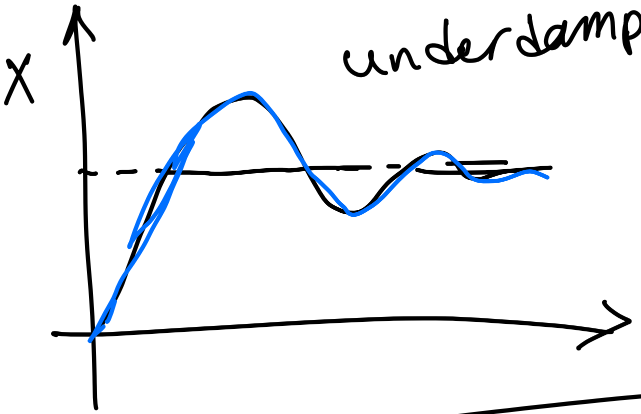
$$\boxed{\frac{C}{2m} = \zeta\,\omega_n}$$

$$\boxed{\frac{K}{m} = \omega_n^2}$$

Standard form of a second order systems

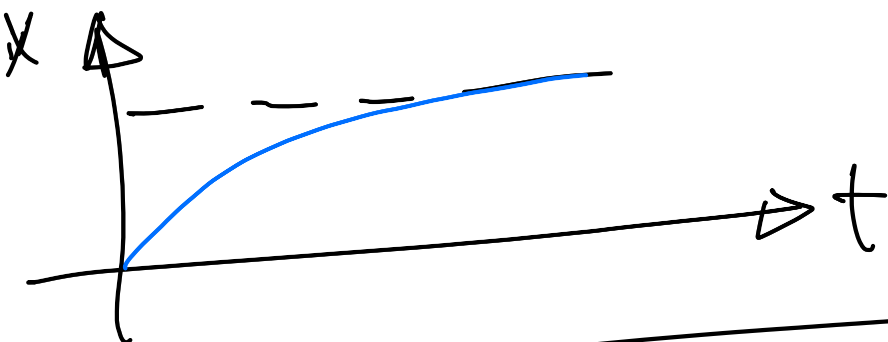
$0 < \xi < 1 \longrightarrow \boxed{\xi = \frac{c}{c_c} < 1 \quad c < c_c}$

underdamped



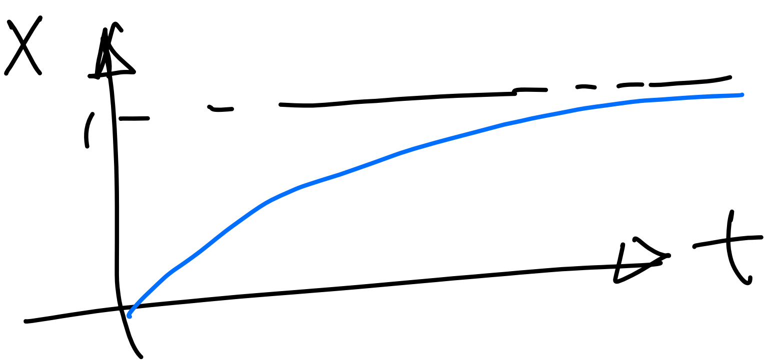
$\xi = 0 \Longrightarrow c = c_c$

critical damping



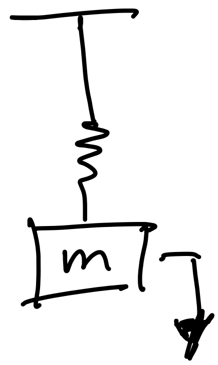
$\xi > 1 \Longrightarrow c > c_c$

overdamped

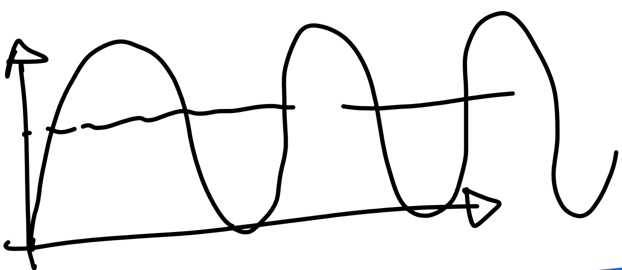


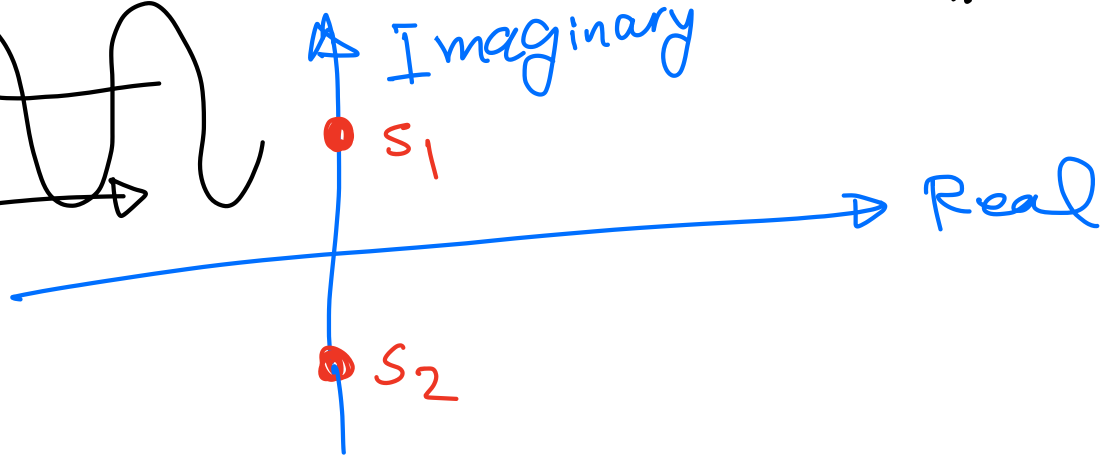
$$\zeta = 0 \implies c = 0$$

$$S_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}$$



$$S_1, S_2 = \pm\sqrt{-\frac{K}{m}} = \pm i\omega_n$$





oscillation Does not stop

$$S_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}$$

$$S_{1,2} = \omega_n\left[-\zeta \pm i\sqrt{1-\zeta^2}\right]$$

$\zeta > 1 \implies$ overdamped

$0 < \zeta < 1 \implies$ underdamped

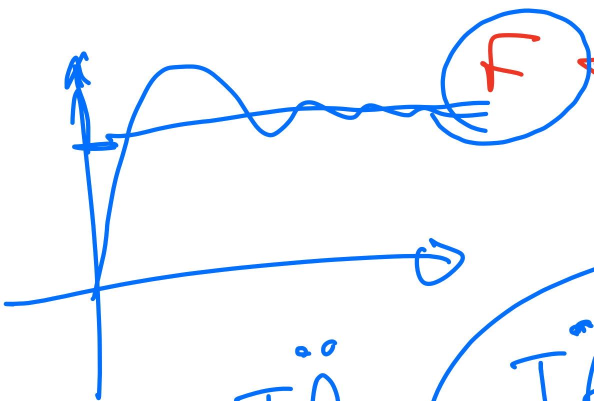
$$\boxed{\begin{array}{l} \omega_n = \sqrt{K/m} \\ \zeta = 2\sqrt{Km} \end{array}}$$

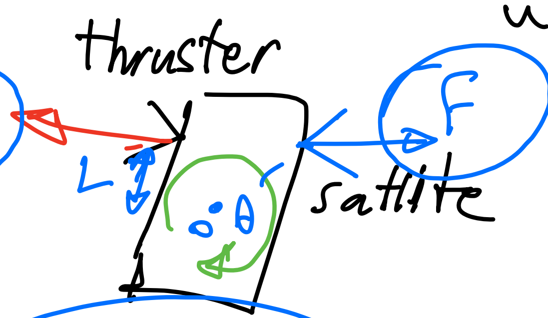
$\zeta = 1 \Rightarrow$ critical Damping

$$\boxed{\zeta = \frac{c}{c_c}}$$

$\zeta = 0 \Rightarrow$ No Damping → cantinuos oscilation

unstable





$$\Sigma M_o = I\ddot{\theta}$$

$$I\ddot{\theta} = FL$$