a

## Energy method :

Conservation of Energy

$$T + U = \text{constant}$$

Kinetic energy

$\frac{1}{2} m v^2$

potential energy

Gravity – spring

~~Friction~~

$$T_1 + U_1 = T_2 + U_2$$

The sum of kinetic and potential

energies at state/instant 1 is equal to the sum of kinetic and potential energy at state/instant 2.

---

Let 1 be the time when the mass is passing through its static equilibrium position and choose $U_1 = 0$ as reference for the potential energy. Let 2 be the time corresponding to the maximum displacement of the mass ($T_2 = 0$)

Therefore:

$$T_1 + 0 = 0 + U_2$$

$U_1 = 0$ $\quad$ $T_2 = 0$

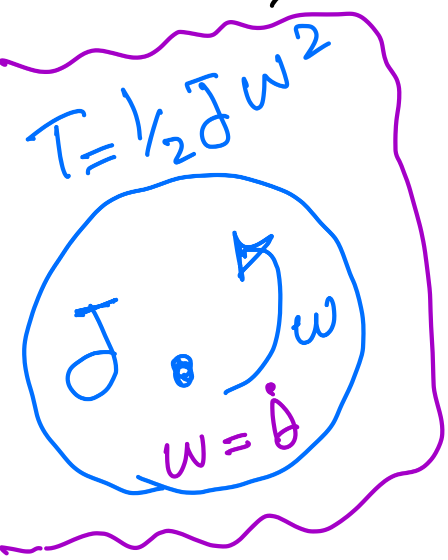
In harmonic motion:

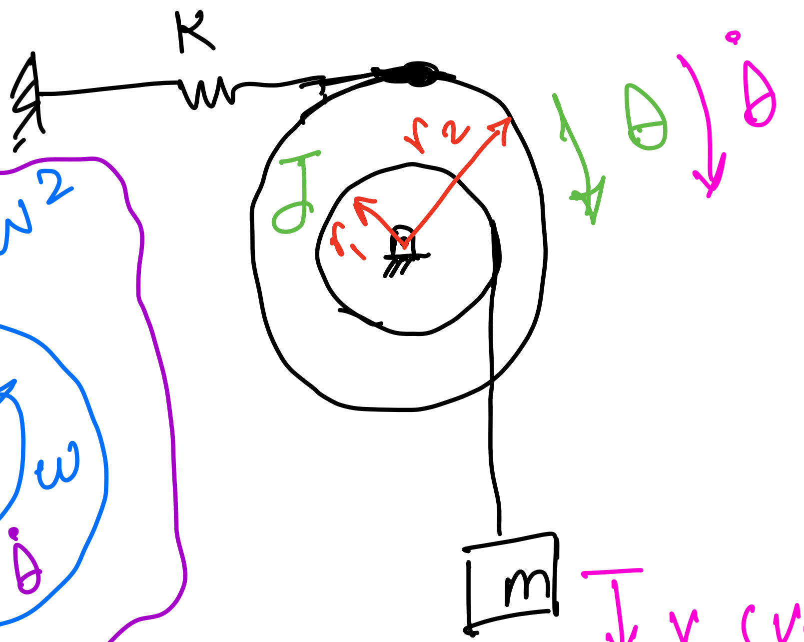
$$T_{max} = U_{max}$$

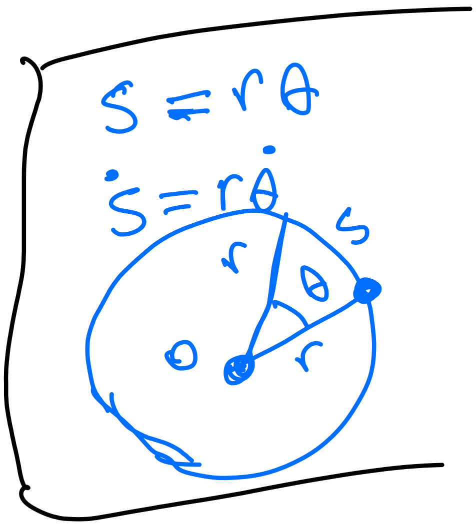
---

Example (2.3-1 Book):

Determine the natural frequency of the system below.




$v$ (velocity) $= r_1 \dot{\theta}$

$$T_{max\ mass(m)} = \frac{1}{2} m v^2 = \frac{1}{2} m (r_1 \dot{\theta})^2$$

$$T_{max\ Disk} = \frac{1}{2} J \dot{\theta}^2$$

$$T_{max} = T_{mass} + T_{Disk} = \left[\frac{1}{2} m (r_1 \dot{\theta})^2 + \frac{1}{2} J \dot{\theta}^2\right]$$

Potential energy of the spring (stored in the spring):

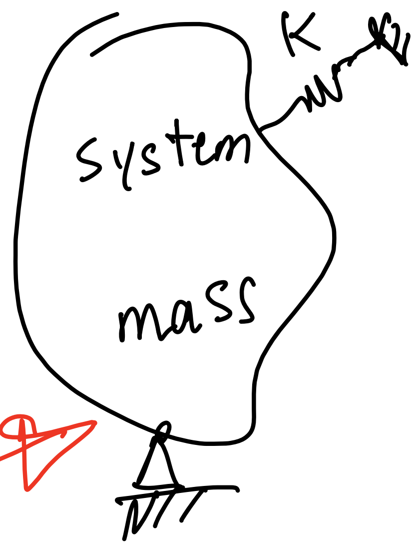
$$U_{max} = \frac{1}{2} K (r_2 \theta)^2$$

$$\frac{1}{2} m (r_1 \dot{\theta})^2 + \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} K (r_2 \theta)^2$$

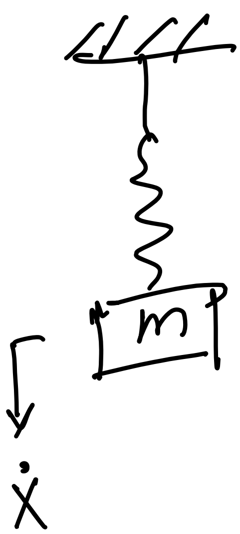
---

## Rayleigh method: Effective mass

Effective mass which can be equivalent to a single mass ($m_{eff}$)




$\equiv$




$m_{eff} = ?$

$$T = \frac{1}{2} m_{eff} \dot{x}^2$$

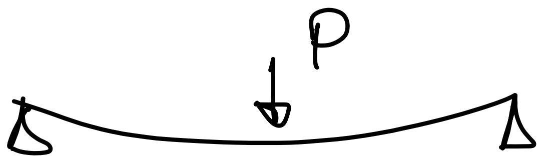
An equivalent lumped mass

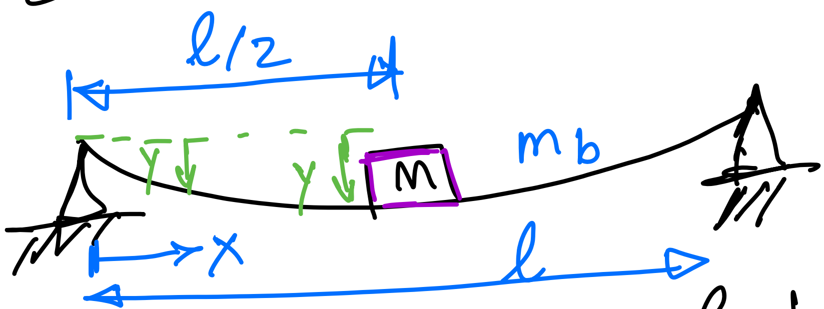
If the stiffness is known, the natural frequency can be calculated by

$$\omega_n = \sqrt{\frac{K}{m_{eff}}}$$

---

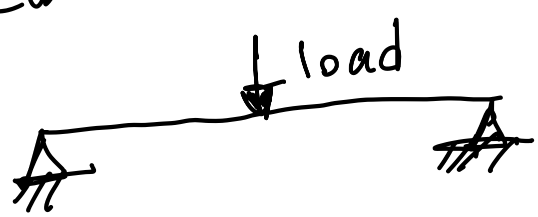
Example 2.4-2




Effective mass of beam

Assume the deflection of the beam is due to a concentraded load at midspan.




$$y = y_{max}\left[\frac{3x}{l} - 4\left(\frac{x}{l}\right)^3\right] \qquad \left(x \leq \tfrac{1}{2} l\right)$$

$$T_{max} = \frac{1}{2}\int_0^{l/2} \frac{2m_b}{l}\left\{\dot{y}_{max}\left[\frac{3x}{l} - 4\left(\frac{x}{l}\right)^3\right]\right\}^2 dx$$

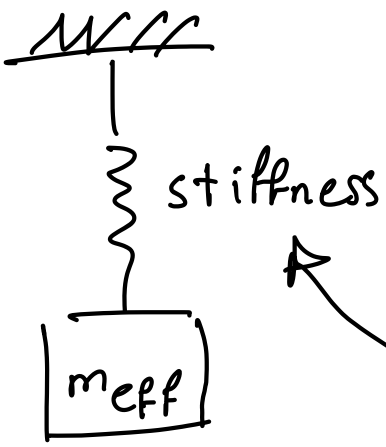
$m_{eff} = M + 0.4857\, m_b$

natural frequency

$$\omega_n = \sqrt{\frac{k}{m_{eff}}}$$

K: stiffness

$$\omega_n = \sqrt{\frac{48EI}{l^3 (M + 0.4857 m_b)}}$$




Midspan deflection due to a concentrated force P applied at Midspan is

$$y_{max} = \frac{Pl^3}{48EI}$$

Force = stiffness × Displacem.

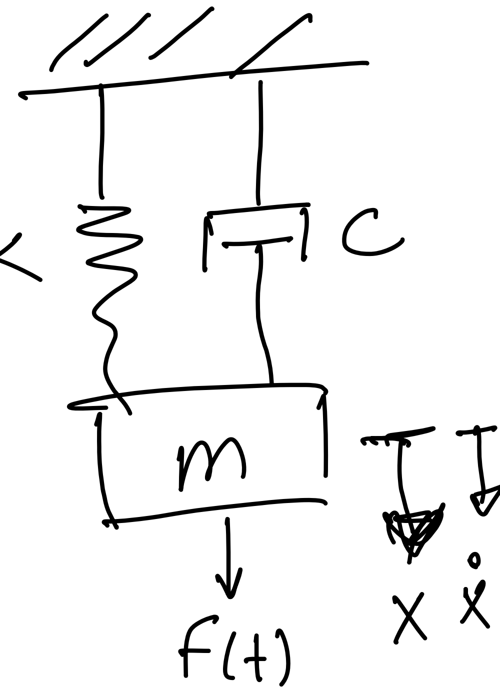
$$P = \underbrace{\frac{48EI}{l^3}}_{\text{stiffness}} y_{max}$$

## Viscously Damped Free vibration

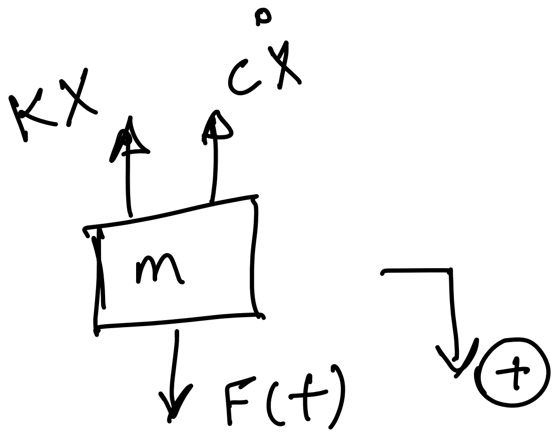
Viscous Damping force is expressd by the equation

$$F_d = c\dot{x}$$

Damping coefficent




F.B.D




$$\downarrow + \quad \sum F_x = m\ddot{x}$$

$$F(t) - c\dot{x} - Kx = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$

[Free vibration
F(t) = 0]

$$F(t) = 0$$

$$m\ddot{x} + c\dot{x} + Kx = 0$$

Traditional approach is to assume a solution of the form:

$$x = e^{st}$$

where s is a constant.

substitute x in the equation:

$$x = e^{st} \longrightarrow \dot{x} = s e^{st} \longrightarrow \ddot{x} = s^2 e^{st}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m s^2 e^{st} + c s e^{st} + k e^{st} = 0$$

$$\underbrace{(ms^2 + cs + k)}_{=0} \; \underset{\neq 0}{e^{st}} = 0$$

$$ms^2 + cs + k = 0$$

$$\rightarrow s^2 + c/m \, s + \frac{k}{m} = 0$$

Solve for s:

$$s_{1,2} = -\frac{c}{m} \pm \underbrace{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}_{\text{Real, Imaginary ?}}$$

underdamped