Example 2.4 (Book)

For the following frame, find the values of missing elements and complete the matrix representation of frame:

$$F = \begin{bmatrix} ? & 0 & ? & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Constraint equations:

1. $\vec{n} \cdot \vec{o} = 0$

$$(n_x \vec{i} + n_y \vec{j} + n_z \vec{K}) \cdot (o_x \vec{i} + o_y \vec{j} + o_z \vec{K}) = 0$$

$$n_x o_x + n_y o_y + n_z o_z = 0$$

$n_x(0) + 0.707(o_y) + n_z(o_x) = 0$

2. $\vec{n} \cdot \vec{a} = 0$

$n_x a_x + n_y a_y + n_z a_z = 0$

$n_x a_x + 0.707 a_y + n_z(0) = 0$

3. $\vec{a} \cdot \vec{o} = 0$

$a_x o_x + a_y o_y + a_z o_z = 0$

$a_x(0) + a_y(o_y) + 0(o_z) = 0$

4. $|\vec{n}| = 1$

$n_x^2 + n_y^2 + n_z^2 = 1$

$n_x^2 + 0.707^2 + n_z^2 = 1$

5. $|\vec{o}| = 1$

$$o_x^2 + o_y^2 + o_z^2 = 1$$

$$0^2 + 0y^2 + 0z^2 = 1$$

6. $|\vec{a}| = 1$

$$a_x^2 + a_y^2 + a_z^2 = 1$$

$$a_x^2 + a_y^2 + 0^2 = 1$$

$$
\begin{cases}
0.707\,o_y + n_z o_z = 0 \\
n_x a_x + 0.707\,a_y = 0 \\
a_y o_y = 0 \\
n_x^2 + n_z^2 = 0.5 \\
o_y^2 + o_z^2 = 1 \\
a_x^2 + a_y^2 = 1
\end{cases}
$$

Two possible solutions:

$$F_1 = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You may not randomly choose one solution

From above.

The same problem may be solved using

$$\vec{n} \times \vec{o} = \vec{a}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ n_x & n_y & n_z \\ o_x & o_y & o_z \end{vmatrix} = a_x\vec{i} + a_y\vec{j} + a_z\vec{K}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ n_x & n_y & n_z \\ o_x & o_y & o_z \end{vmatrix}$$

$$= \vec{i}\,(n_y o_z - n_z o_y)$$

$$- \vec{j}\,(n_x o_z - n_z o_x)$$

$$+ \vec{K}\,(n_x o_y - n_y o_x)$$

$$= a_x \vec{i} + a_y \vec{j} + a_z \vec{K}$$

$$\begin{cases} n_y o_z - n_z o_y = a_x \\ -(n_x o_z - n_z o_x) = a_y \\ n_x o_y - n_y o_x = a_z \end{cases}$$

substituting the values into this equation:

$$0.707\, o_z - n_z o_y = a_x$$

$$-n_x o_z = a_y$$

$$n_x o_y = 0$$

The solution will give F_1 as above.

Example 2.5 (Book)

Find the missing elements of the following frame representation:

$$F = \begin{bmatrix} ? & 0 & ? & 3 \\ 0.5 & ? & ? & 9 \\ 0 & ? & ? & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_x^2 + (0.5)^2 + (0)^2 = 1$$

$$n_x^2 + 0.25 = 1$$

$$n_x = 0.866$$

$$\vec{n} \cdot \vec{o} = 0$$

$$(0.866)(0) + (0.5)(o_y) + (0)(o_z) = 0$$

$$o_y = 0$$

$$|\vec{o}| = 1 \implies o_x^2 + o_y^2 + o_z^2 = 1$$

$$0 + 0 + o_z^2 = 1$$

$$o_z = 1$$

$$\vec{n} \times \vec{o} = \vec{a}$$

$$\vec{i}(0.5) - \vec{j}(0.866) + \vec{k}(0)$$

$$= a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

$a_x = 0.5$ $\quad a_y = -0.866$ $\quad a_z = 0$

Transformation Matrices
(section 2.5 Book)

$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Representation of transformations</u>

A transformation is defined as making a movement in space.

when a frame (a vector, an object, or moving frame) moves in space relative to a fixed reference frame, we represent this motion in a form similar to a frame representation.

The transformation changes the state of a frame represinting the change in its location and orientation.

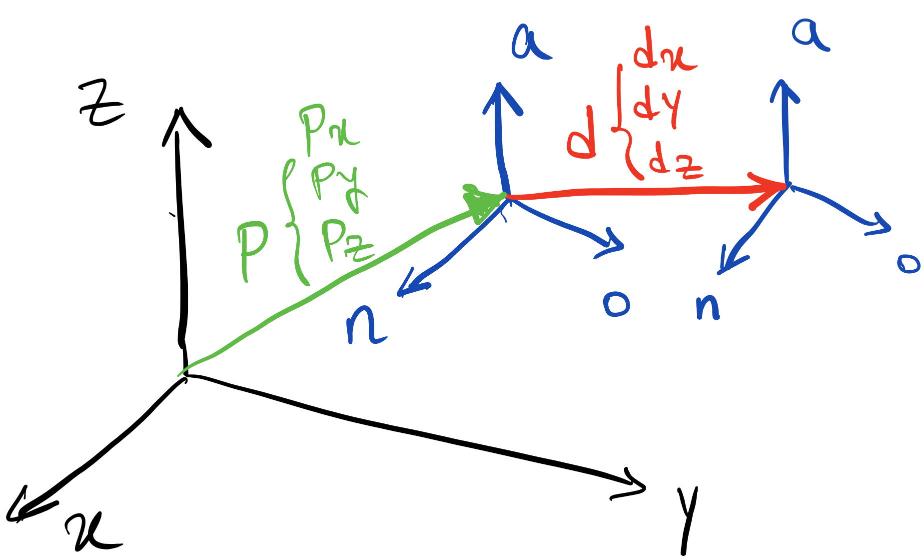
A transformation may be in of of the following forms:

- A pure translation
- A pure rotation about an axis
- A combination of translations and/or rotations

Representation of a pure translation

since the directional vectors do not change in a pure translation, the transformation T will simply be:

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The new location of the frame will be:

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & o_x & a_x & p_x + d_x \\ n_y & o_y & a_y & p_y + d_y \\ n_z & o_z & a_z & p_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This equation is also symbolically written as:

$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old}$$

Example 2.6 (Book)

A frame F has been moved 10 units along the y-axis and 5 units along the z-axis of the reference frame. Find the new location of the frame.

$$F = \begin{bmatrix} 0.5 & -0.5 & 0.6 & 5 \\ 0.3 & 0.8 & 0.4 & 3 \\ -0.7 & 0 & 0.6 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old}$$

$$= Trans(0, 10, 5) \times F_{old}$$

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5 & -0.5 & 0.6 & 5 \\ 0.3 & 0.8 & 0.4 & 3 \\ -0.7 & 0 & 0.6 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0.6 & 5 \\ 0.3 & 0.8 & 0.4 & 13 \\ -0.7 & 0 & 0.6 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation of a pure rotation about an axis (section 2.6.2 Book)