Instrumentation and Controls

ETM 3301

Lecture 26

Instructor

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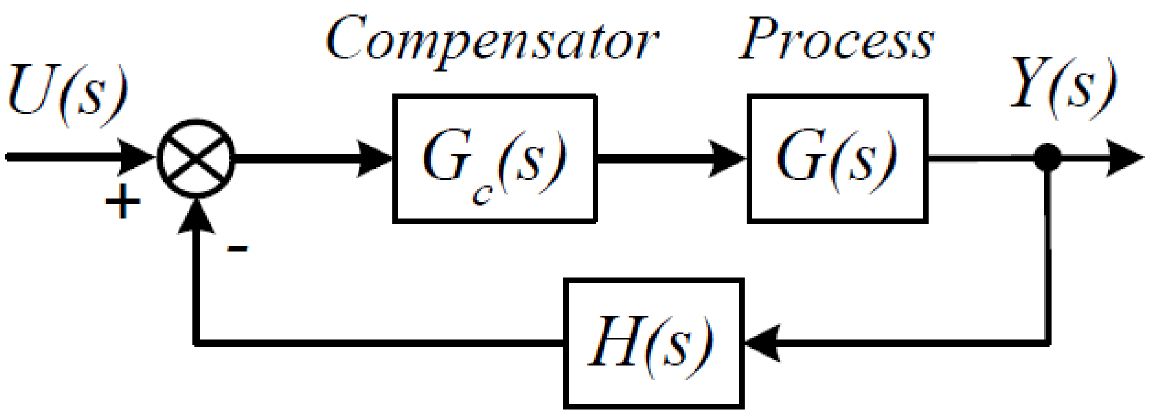
Cascade Compensation

- *Control system design:* the arrangement of the system structure and the selection of suitable components and parameters.
- *Compensation:* the alteration or adjustment of a control system in order to provide the required performance.
- *Compensator:* is an additional component that is inserted into a control system to compensate for a deficient performance.



The insertion of a compensator can increase Phase Margin.

Phase Lead Compensator

Transfer function

$$G_c(s) = \frac{1+\alpha Ts}{1+Ts} \quad (\alpha > 1)$$

$$G_c(j\omega) = \frac{1+\alpha Tj\omega}{1+Tj\omega}$$

$$\left|G_c(j\omega)\right| = \frac{\sqrt{1+(\alpha T\omega)^2}}{\sqrt{1+(T\omega)^2}}$$

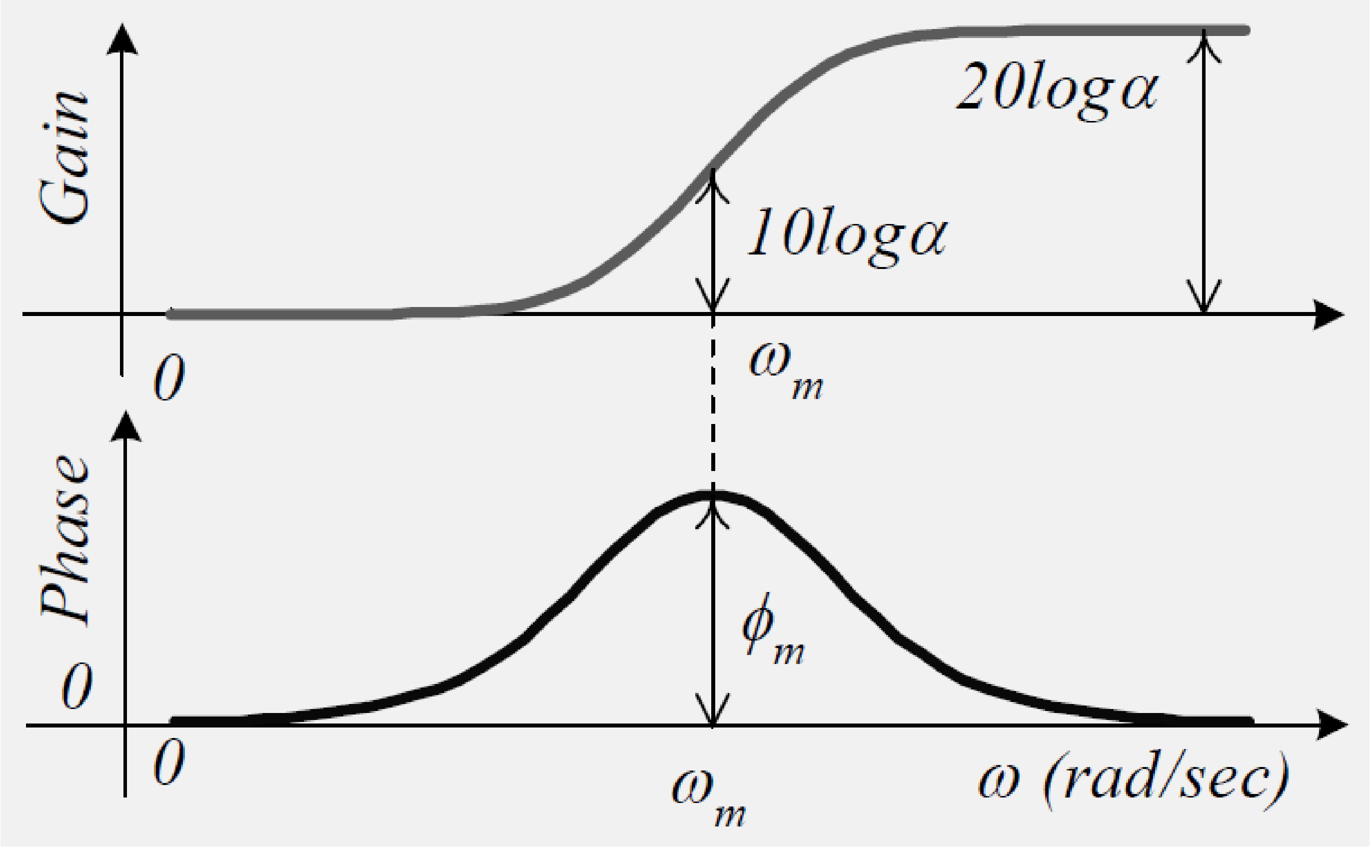
$$\angle G_c(j\omega) = \tan^{-1}\alpha T\omega - \tan^{-1}T\omega > 0$$

$$gain = 20\log_{10}\left|G_c(j\omega)\right| = 20\log_{10}\frac{\sqrt{1+(\alpha T\omega)^2}}{\sqrt{1+(T\omega)^2}}$$

$$= 20\log_{10}\sqrt{1+(\alpha T\omega)^2} - 20\log_{10}\sqrt{1+(T\omega)^2}$$

$$= 10\log_{10}\left(1+(\alpha T\omega)^2\right) - 10\log_{10}\left(1+(T\omega)^2\right)$$

Phase Lead Compensator



- The maximum value of phase lead occurs at the frequency ω_m:

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

Phase Lead Compensator

- At the frequency ω_m, the maximum phase is ϕ_m

$$\sin\phi_m = \frac{\alpha - 1}{\alpha + 1}$$

$$\textit{gain is} \quad \sqrt{\alpha} \quad \textit{or} \quad 10\log_{10}\alpha$$

- *The bigger the value of α, the bigger the maximum phase lead ϕ_m.*

Function of Phase Lead Compensator

- ***Compensated system OLTF:*** $G_c(s)G(s)$

Gain $\quad \left|G_c(j\omega)\right| \times \left|G(j\omega)\right| \quad 20\log\left|G_c(j\omega)\right| + 20\log\left|G(j\omega)\right|$

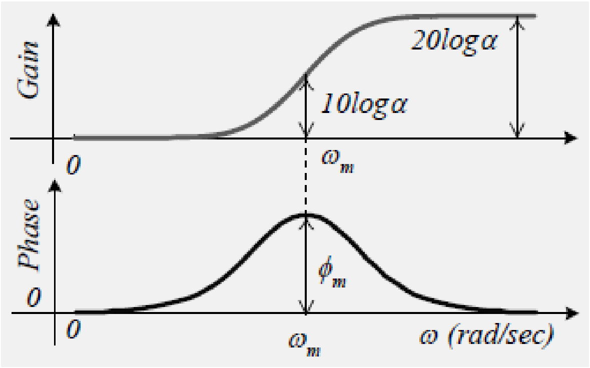
Phase $\quad \angle G_c(j\omega) + \angle G(j\omega)$

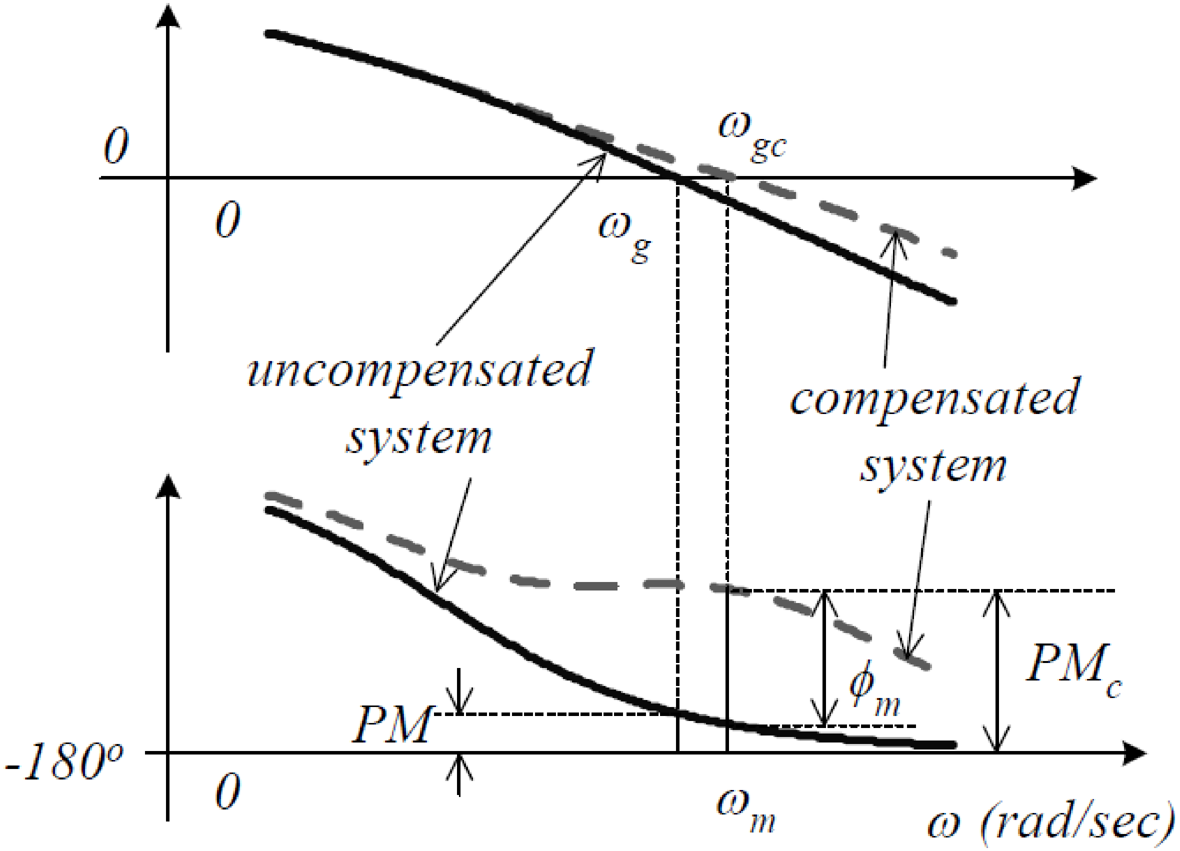
- Both gain and phase are increased, *but the increase in the phase is more significant.*
- The effect of a phase lead compensator when introduced into the forward path of a closed-loop control system:
  - to produce a faster and stable system by increasing the phase margin!

Phase lead compensator design principle

- To generate the maximum increase in the phase margin, the biggest phase lead (ϕ_m) must occur at the gain crossover frequency of the compensated system.

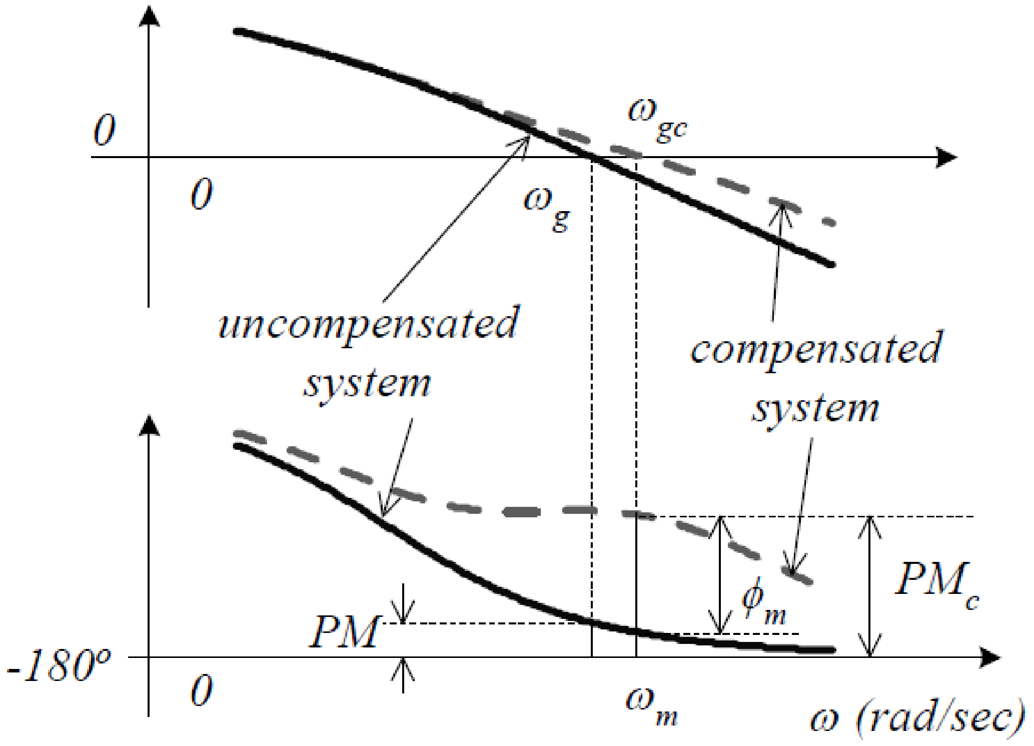
$$\omega_m = \omega_{gc}$$
$$(0db\ \ frequency)$$





Phase lead compensator design principle

$$\omega_m = \omega_{gc}$$
$$(0db\ \ frequency)$$



compensated gain =uncompensated gain+ compensator gain

at ω_{m} compensator gain $= 10\log\alpha$ (dB)

uncompensated gain must be -10logα (dB) at ω_{m}

Phase lead compensator design procedure

- Adjust the system parameter (K) to satisfy the steady state error requirement.
- Plot Bode diagram of the uncompensated system frequency response.
- Evaluate uncompensated system phase margin.
- Allowing for a small amount of safety (≈*10%-30%*), determine the necessary phase lead ϕ_m.
- Find α using equation:

$$\sin\phi_m = \frac{\alpha - 1}{\alpha + 1} \quad \Rightarrow \quad \alpha = \frac{1 + \sin\phi_m}{1 - \sin\phi_m}$$

$$G_c(s) = \frac{1 + \alpha T s}{1 + T s} \quad (\alpha > 1)$$

Phase lead compensator design procedure

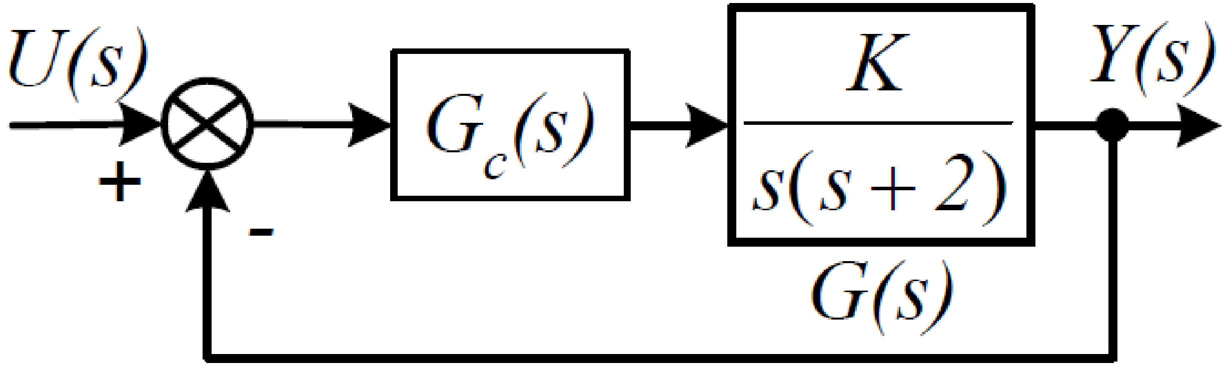
- Evaluate $10\log\alpha$ and determine the frequency where the uncompensated gain (magnitude) curve is equal to $-10\log\alpha$ dB. Because the compensator provide a gain of $10\log\alpha$ at ω_m, this frequency is the new 0 dB crossover frequency and ω_m simultaneously.
- Find the value of T using equation:

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad T = \frac{1}{\omega_m\sqrt{\alpha}}$$

- Plot the compensated system frequency response and check the phase margin, repeat the design if necessary.

Phase-Lead Compensator Design Example

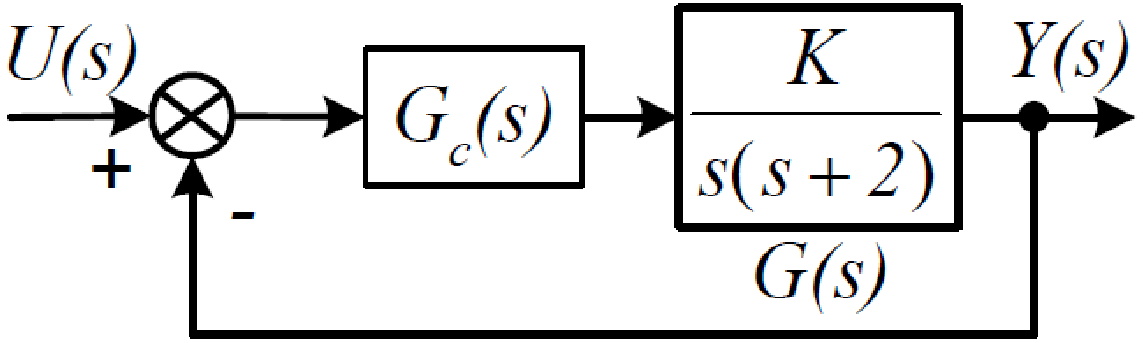
- For the system with the following block diagram



a) find the parameter K to make the system have 5% steady state error for an unit ramp input.

b) design the compensator $G_c(s)$ to provide the phase margin of $45°$.

Phase-Lead Compensator Design Example



- The steady state error for a unit ramp input is:

$$e_{ss} = \frac{1}{K_V} \qquad K_V = \lim_{s \to 0} s\left[G_c(s)G(s)\right]$$

$$K_V = \lim_{s \to 0} s\left[\frac{1+\alpha Ts}{1+Ts}\frac{K}{s(s+2)}\right] = \frac{K}{2}$$

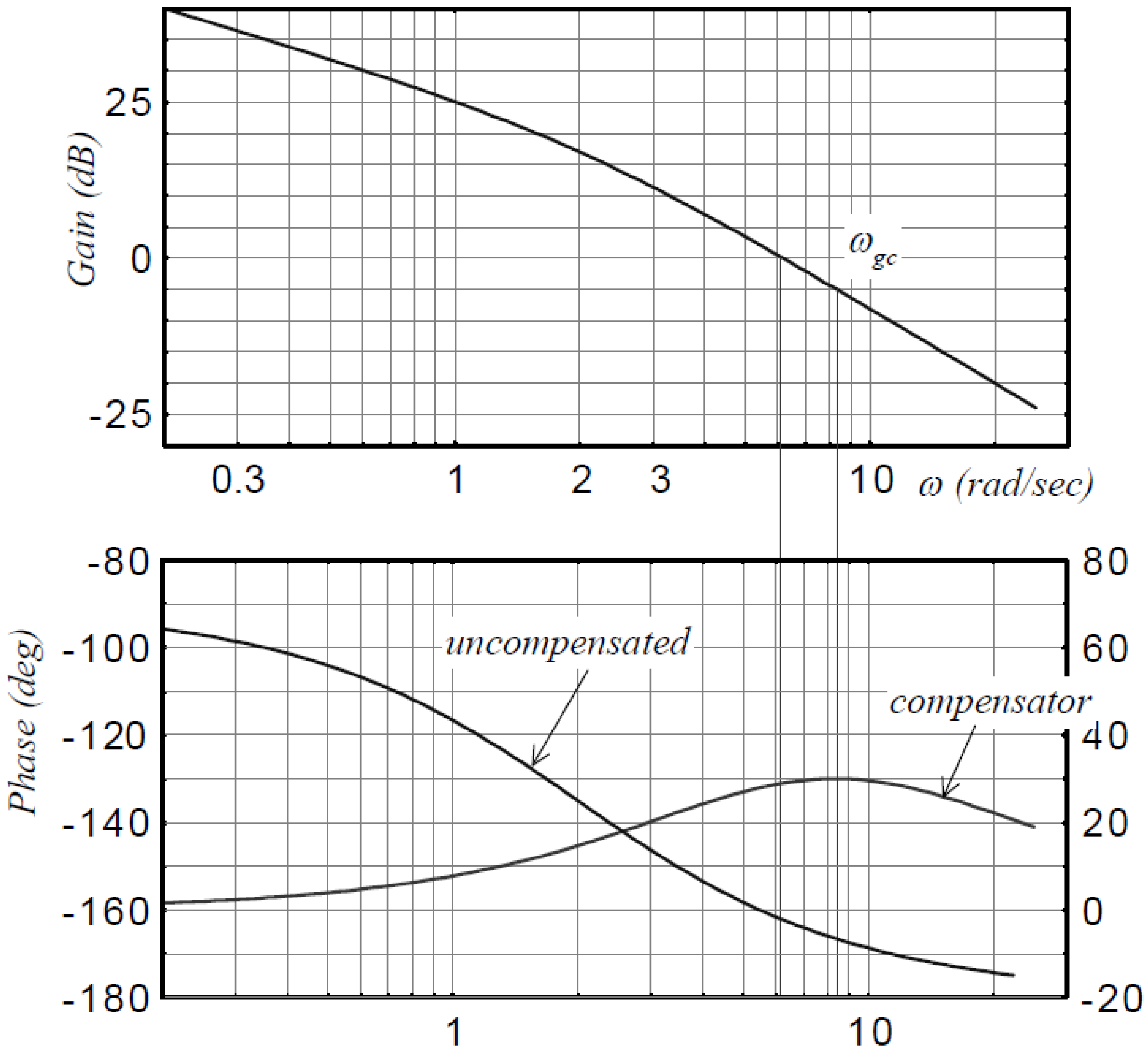
$$e_{ss} = \frac{2}{K} = 0.05 \quad \Rightarrow \quad K = 40 \qquad\qquad G(s) = \frac{40}{s(s+2)}$$

Phase-Lead Compensator Design Example

G(jω)

gain crossover frequency $\omega_g = 6.2\ rad/s$

phase margin $PM = 180° - 162° = 18°$



Phase-Lead Compensator Design Example

The required maximum phase lead is:

$$\phi_m = 45° - 18.2° + 3° \approx 30°$$

Design safety margin

$$\frac{\alpha - 1}{\alpha + 1} = \sin\phi_m = \sin 30° = 0.5 \Rightarrow \alpha = 3$$

The maximum phase lead occurs at ω_m, and this frequency will be selected so that the new gain crossover frequency ω_{gc} and ω_m coincide.

The compensator gain at ω_m is $10\log\alpha = 4.8dB$.

Phase-Lead Compensator Design Example

The compensated gain crossover frequency ω_{gc} is then evaluated where the gain of the uncompensated FR $G(j\omega)$ is $-4.8dB$.

From Bode diagram, it can be found that this frequency is:

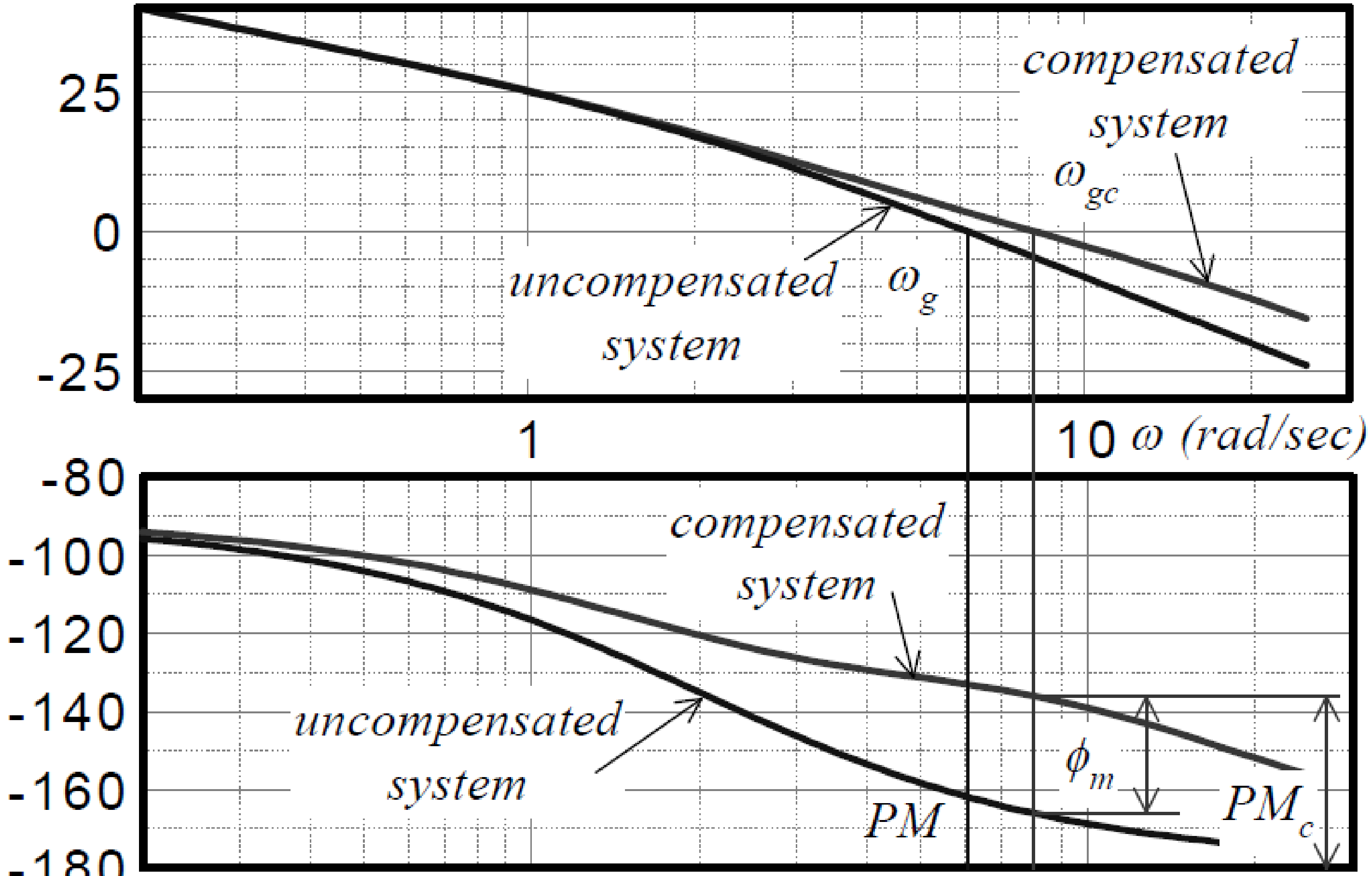
$$\omega_m = \omega_{gc} = 8.4$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad T = \frac{1}{\omega_m\sqrt{\alpha}} = 0.0687$$

Finally, the compensator is $G_c(s) = \dfrac{1+\alpha Ts}{1+Ts} = \dfrac{1+0.2062s}{1+0.0687s}$

Compensated system OLTF $G_c(s)G(s) = \dfrac{20(1+0.2062s)}{s(1+0.5s)(1+0.0687s)}$

Phase-Lead Compensator Design Example



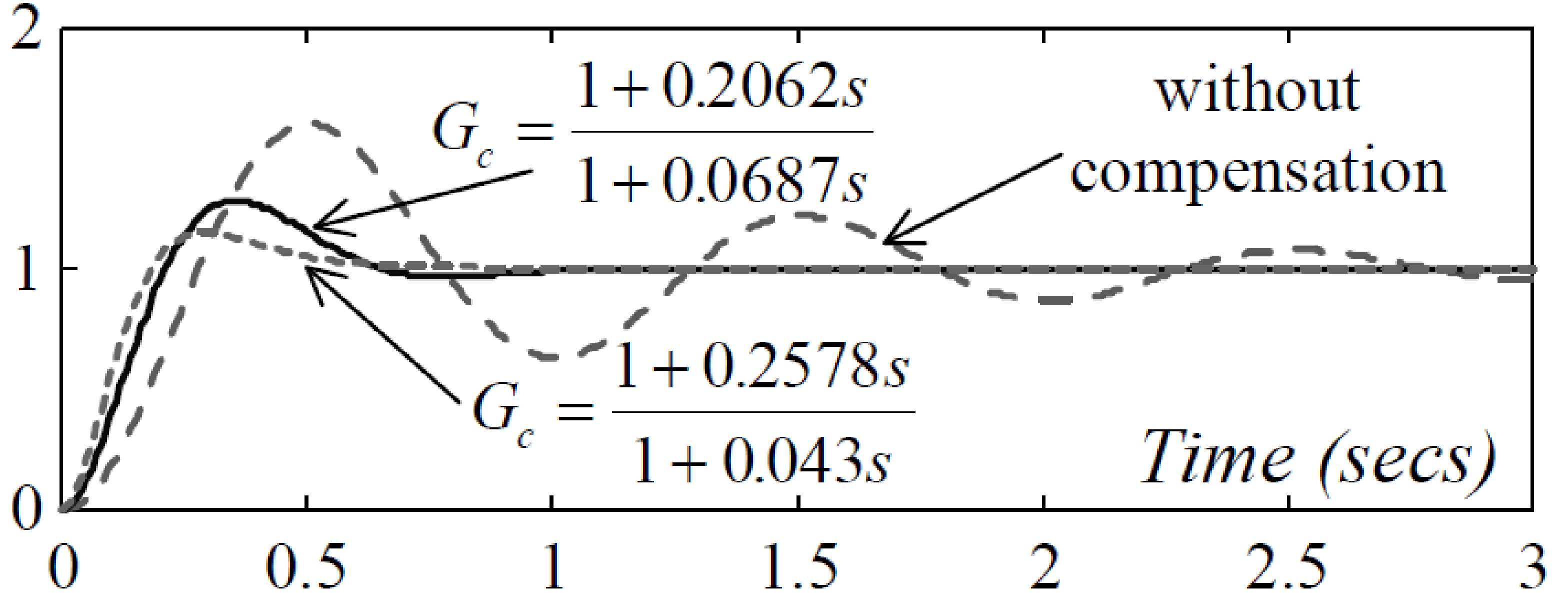
Compensated system phase margin PM_c=43.8°

Design requirement is not satisfied.

Phase-Lead Compensator Design Example

Further improvement can be made by increasing ϕ_m to *45°*.

$$G_c = \frac{1+0.2578s}{1+0.043s} \qquad PMc = 57 \qquad T = 0.043 \qquad \alpha = 6$$

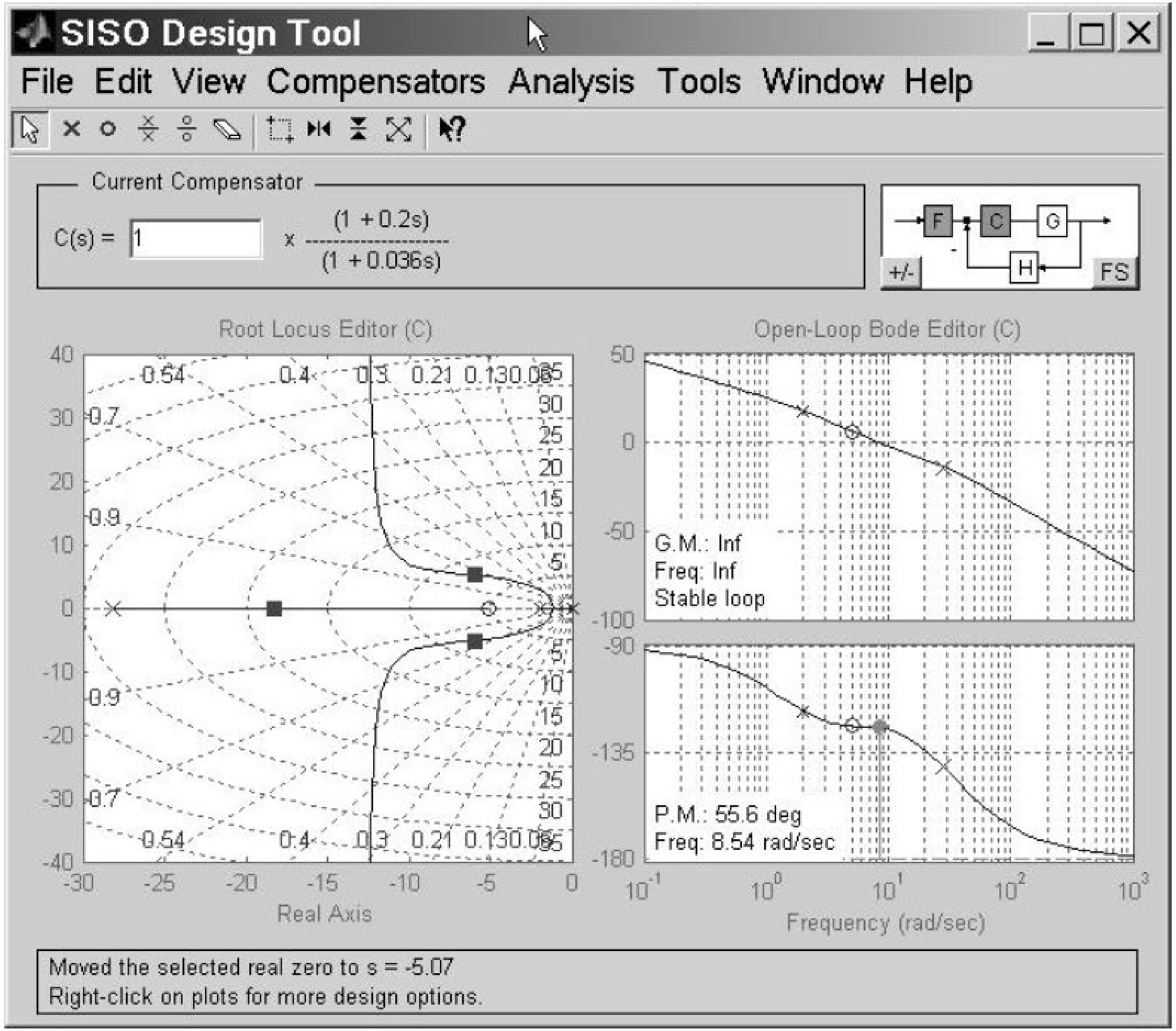


Unit step responses of closed-loop systems.

Matlab Phase-Lead Compensator

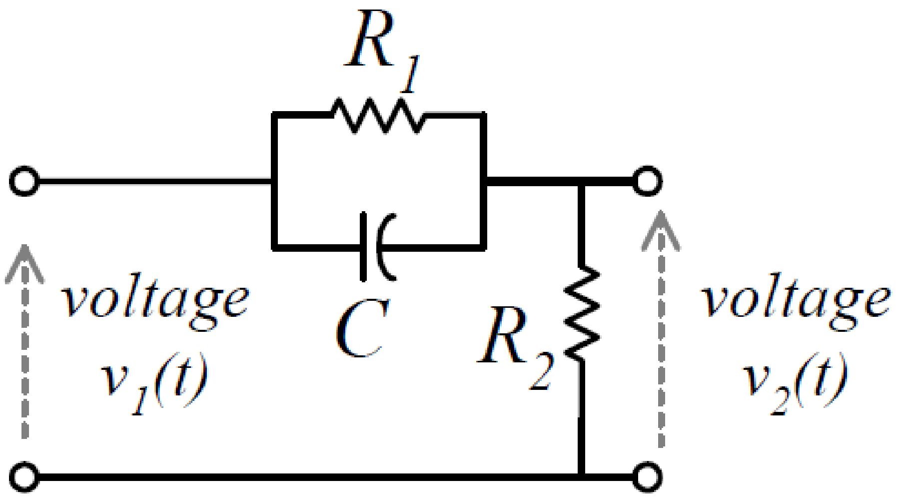
```
>>N=40;

>>D=1 2 0];

>>G=tf(N,D)
```

```
>>sisotool
```



Physical implementation of phase-lead compensator

The phase-lead compensator can be implemented using an electric network.



$$G_N(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2}{R_2 + \{R_1(1/Cs)/[R_1 + 1/Cs]\}}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right)\frac{(1 + R_1 Cs)}{1 + \frac{R_1 R_2 C}{R_1 + R_2} s}$$

Physical implementation of phase-lead compensator

$$G_N(s) = \left(\frac{R_2}{R_1 + R_2}\right)\frac{(1 + R_1 C s)}{1 + \frac{R_1 R_2 C}{R_1 + R_2} s}$$

Define:

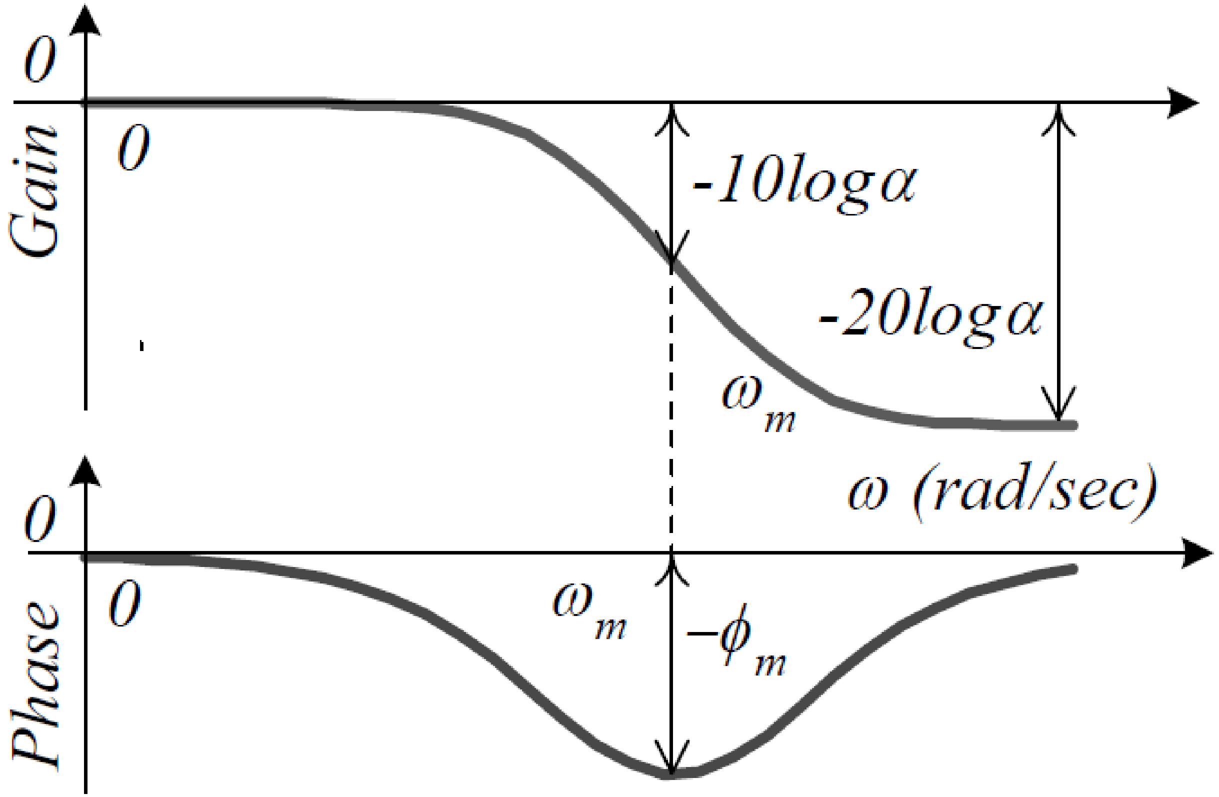
$$\alpha = \frac{R_1 + R_2}{R_2}$$

$$T = \frac{R_1 R_2 C}{R_1 + R_2}$$

$$G_N(s) = \left(\frac{1}{\alpha}\right)\left(\frac{1 + \alpha T s}{1 + T s}\right)$$ phase lead compensator

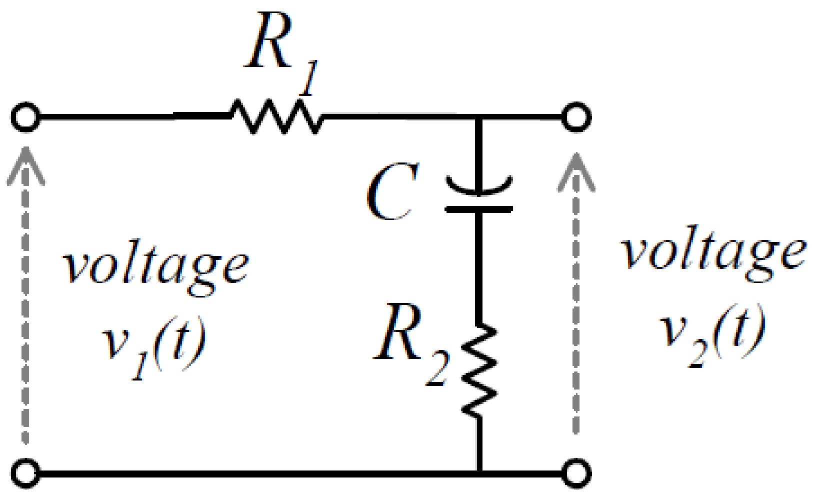
Phase lag Compensator

Transfer function $$G_c(s) = \frac{1+Ts}{1+\alpha Ts} \quad (\alpha > 1)$$



Phase Lag Features

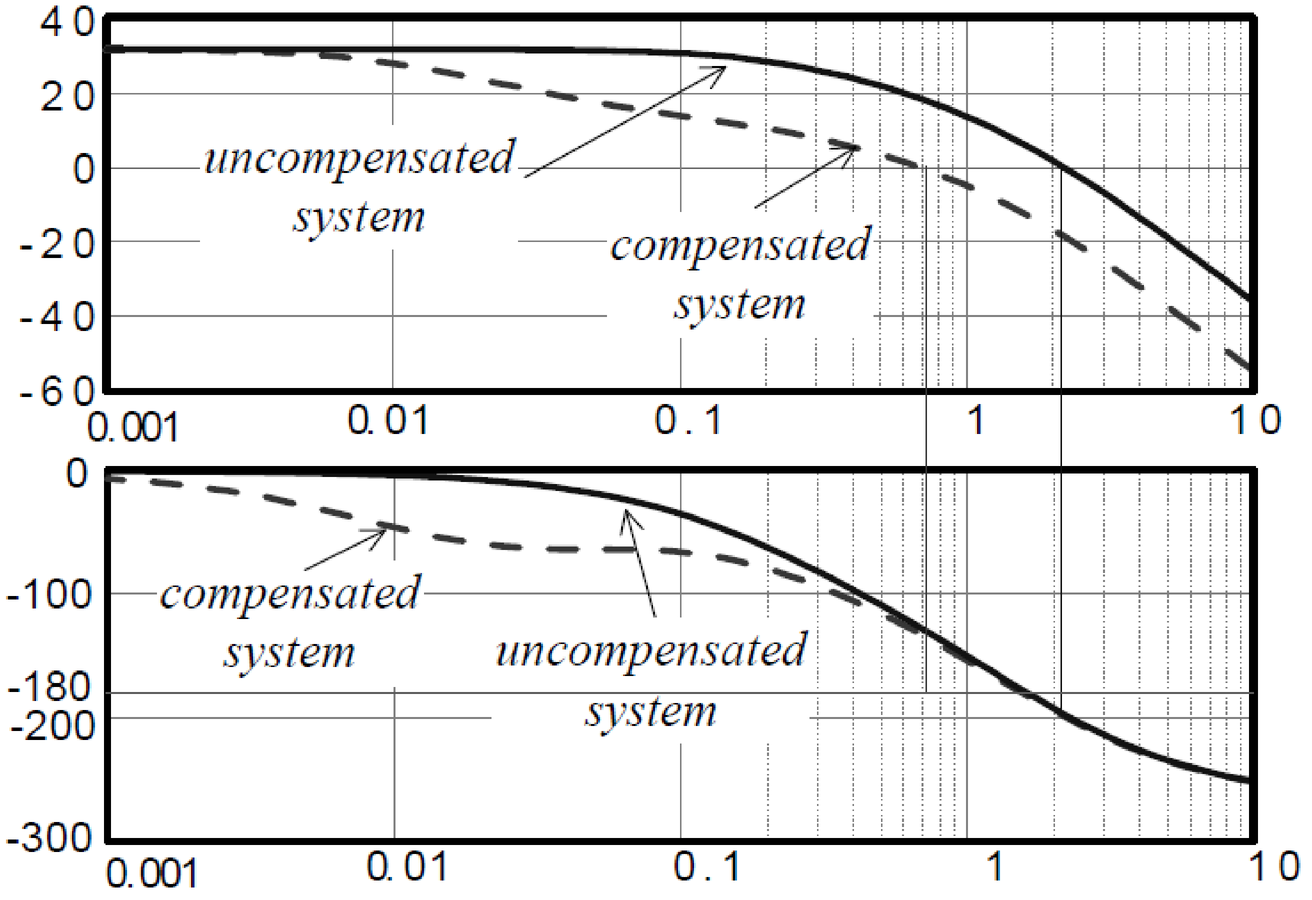
- It is a low-pass filter.
- The low frequency gain can be raised to improve the steady-state error performance.
- The high frequency gain is reduced to yield stability.
- Use it as a compensator, the gain crossover frequency can be reduced and hence the phase margin increased.



Phase-Lag Compensator Design Example

$$G(s) = \frac{40}{(1+5s)(1+s)(1+0.5s)}$$

Uncompensated system is unstable.



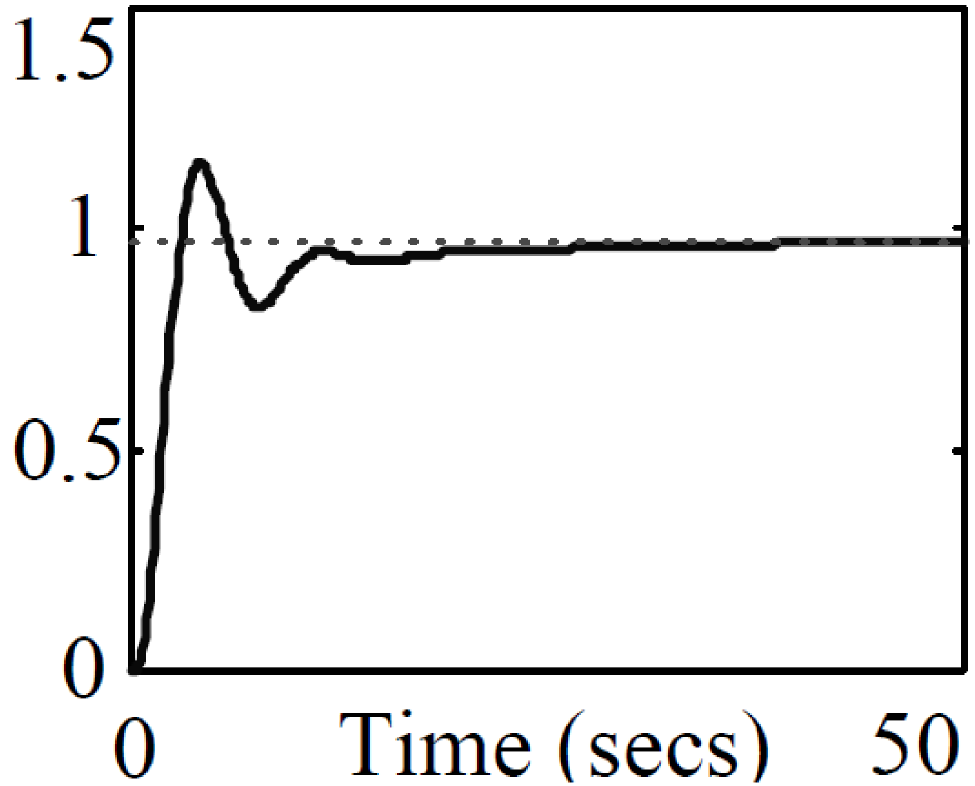
By reducing gain crossover frequency, the phase margin is increased.

Phase-Lag Compensator Design Example

Phase lag compensator $$G_c(s) = \frac{1+14s}{1+8.6\times 14s}$$

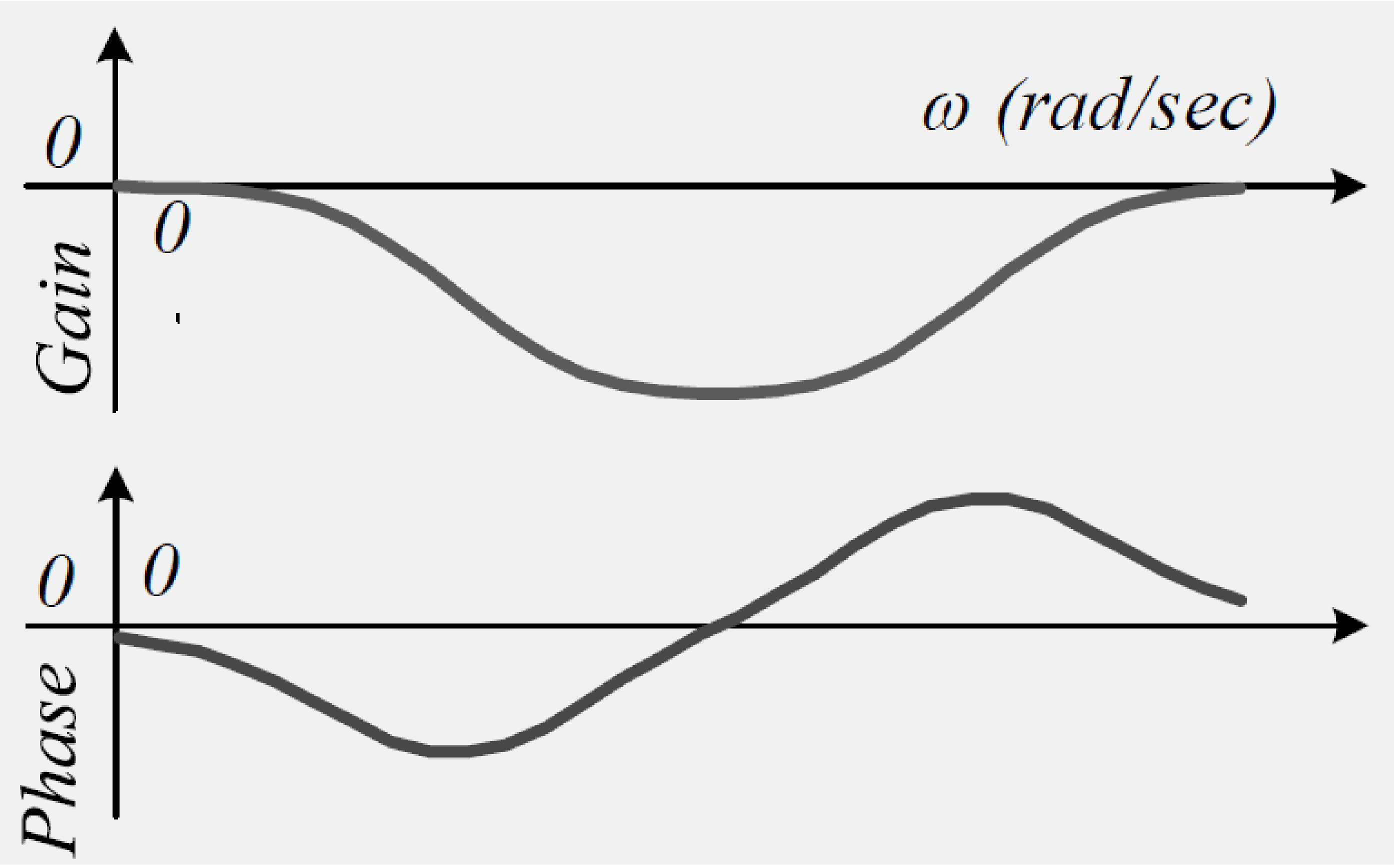
system is stabilised

step response is satisfactory



Lag-Lead compensation

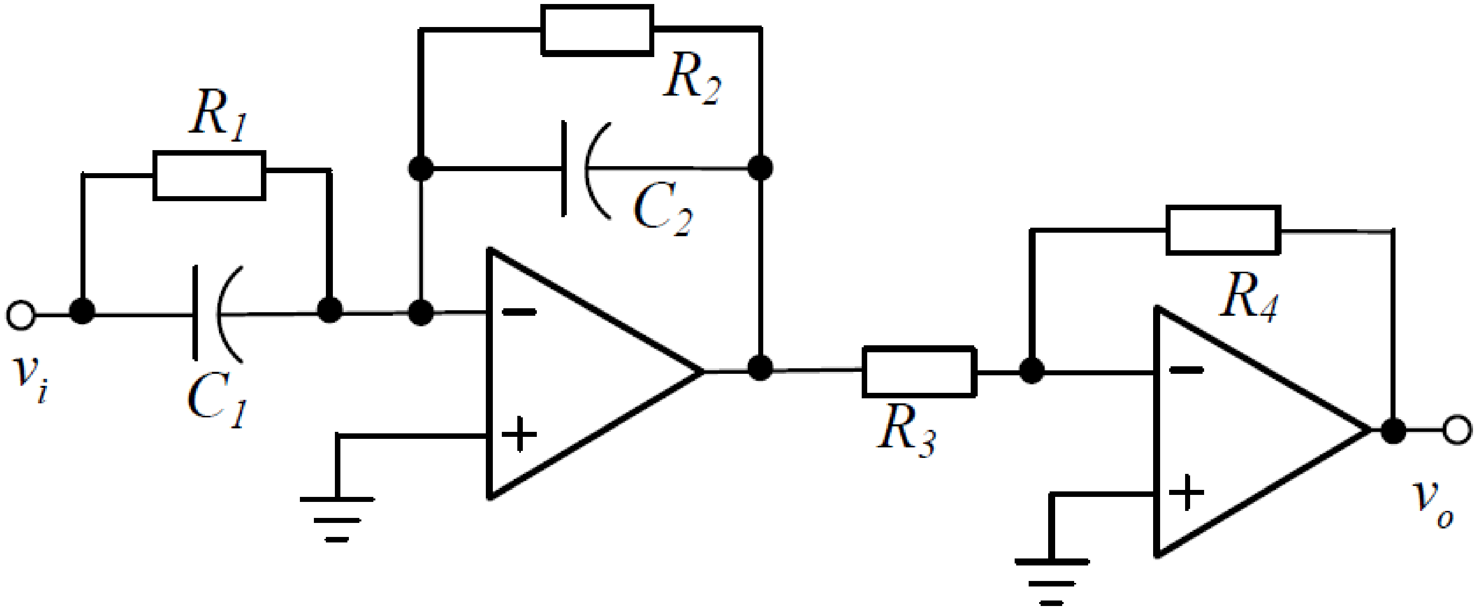
$$G_c(s) = \frac{1+T_1 s}{1+\beta T_1 s}\frac{1+\alpha T_2 s}{1+T_2 s} \qquad \alpha > 1,\ \beta > 1,\ T_1 > T_2$$



Lag-Lead compensation

- Combines the advantages of phase-lag and phase-lead compensators.
- First, the lag compensator is designed to yield the proper steady-state performance with the improved stability.
- Next, the lead compensator is designed to speed up the transient response.

Implement Phase Lead/Lag Compensator using Operational Amplifiers



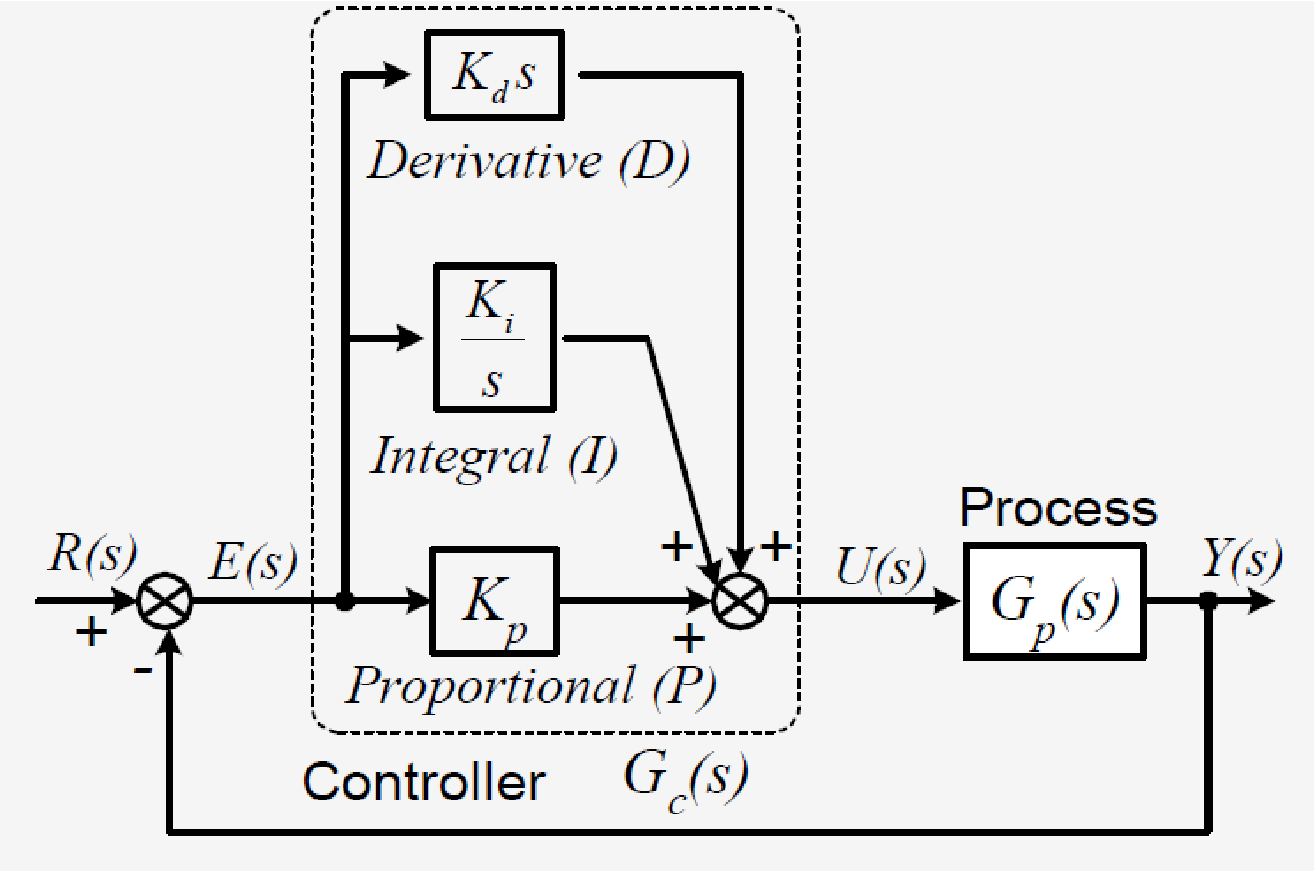
$$\frac{V_o(s)}{V_i(s)} = \left(-\frac{\dfrac{R_2}{1+R_2C_2s}}{\dfrac{R_1}{1+R_1C_1s}} \right)\left(-\frac{R_4}{R_3} \right) = \frac{R_4R_2(1+R_1C_1s)}{R_3R_1(1+R_2C_2s)}$$

$R_1C_1 > R_2C_2$: phase lead $\qquad$ $R_1C_1 < R_2C_2$: phase lag

Three-term (PID) Controller

- Controller $G_c(s)$ is used to improve the closed-loop system performance.

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$



PD Control & Phase-Lead Compensator

- *Proportional* (P) + *Derivative (D) Controller*.

$$G_c(s) = K_p + K_d s$$

Phase-Lead Compensator $\longrightarrow$ $G_c(s) = K\dfrac{1+\alpha Ts}{1+Ts} \quad (\alpha > 1)$

when T is very small and αT is in a reasonable size

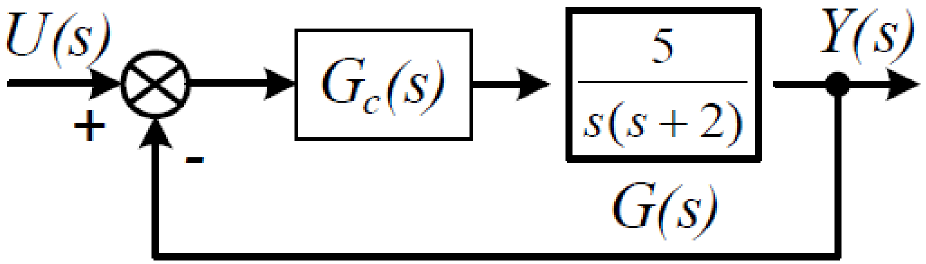
$$G_c(s) = K\frac{1+\alpha Ts}{1+Ts} \approx K\frac{1+\alpha Ts}{1} = K + K\alpha Ts = K_p + K_d s$$

PD controller is an extreme case of phase-lead compensator

Disk drive control system improvement – PD controller

Design requirements: for a unit step input

$$e_{ss}=0 \quad PO \leq 2\% \quad t_s \leq 0.08s$$



$$G_c(s) = K_p + K_d s$$

$$T(s) = \frac{5\dfrac{K_p + K_d s}{s(s+20)}}{1 + 5\dfrac{K_p + K_d s}{s(s+20)} \times 1} = \frac{5K_p + 5K_d s}{s(s+20) + 5K_p + 5K_d s}$$

$$T(s) = \frac{5K_p + 5K_d s}{s^2 + (20 + 5K_d)s + 5K_p}$$

Disk drive control system improvement – PD controller

$$e_{ss}=0 \quad PO\leq 2\% \quad t_s \leq 0.08s$$

$$PO = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.02 \Rightarrow \zeta = \sqrt{\frac{(\ln(0.02))^2}{(\ln(0.02))^2+\pi^2}} = 0.7797$$

$$t_s = \frac{4}{\zeta\omega_n} = 0.08 \Rightarrow \omega_n = \frac{4}{\zeta t_s} = 64.127$$

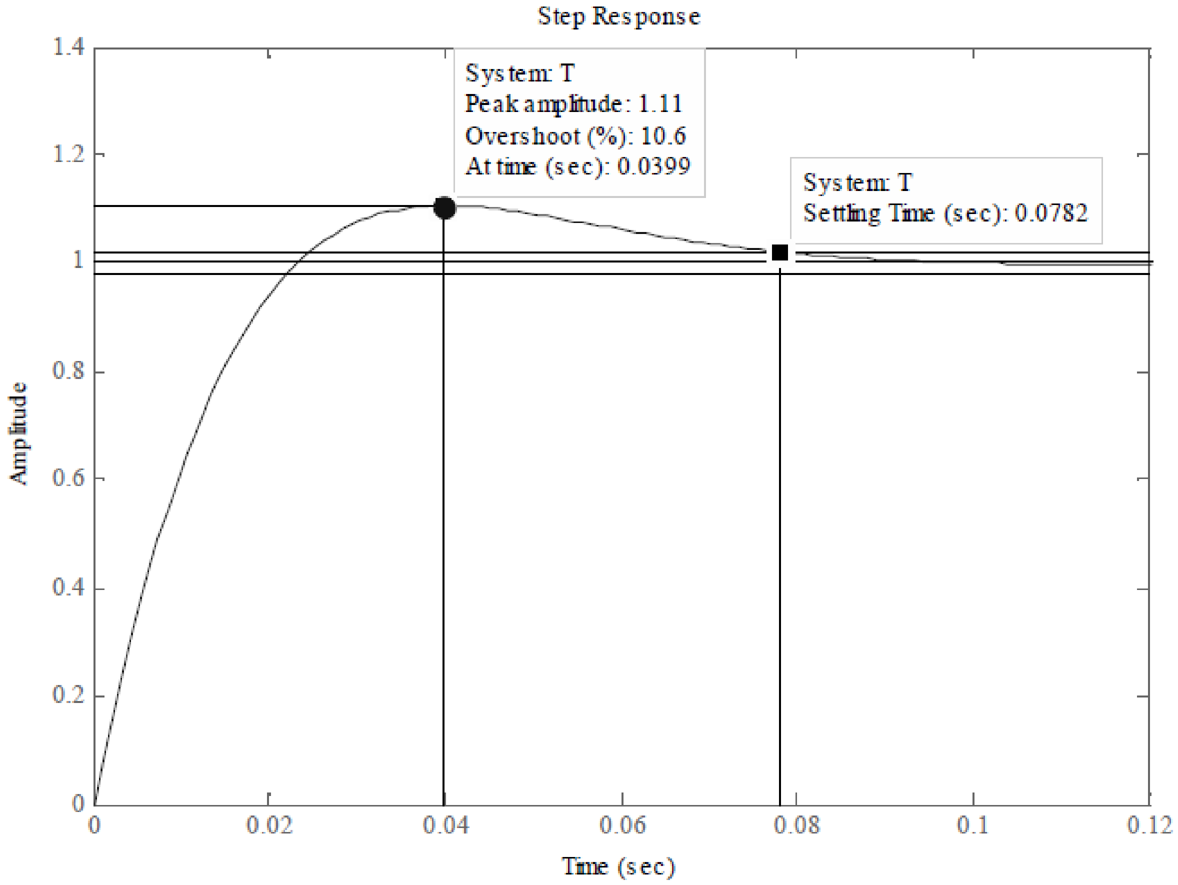
$$s^2 + (20+5K_d)s + 5K_p = 0 \Leftrightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$5K_p = \omega_n^2 \Rightarrow K_p = \frac{\omega_n^2}{5} = 822.45$$

$$20+5K_d = 2\zeta\omega_n \Rightarrow K_d = \frac{2\zeta\omega_n - 20}{5} = 16$$

Disk drive control system improvement – PD controller

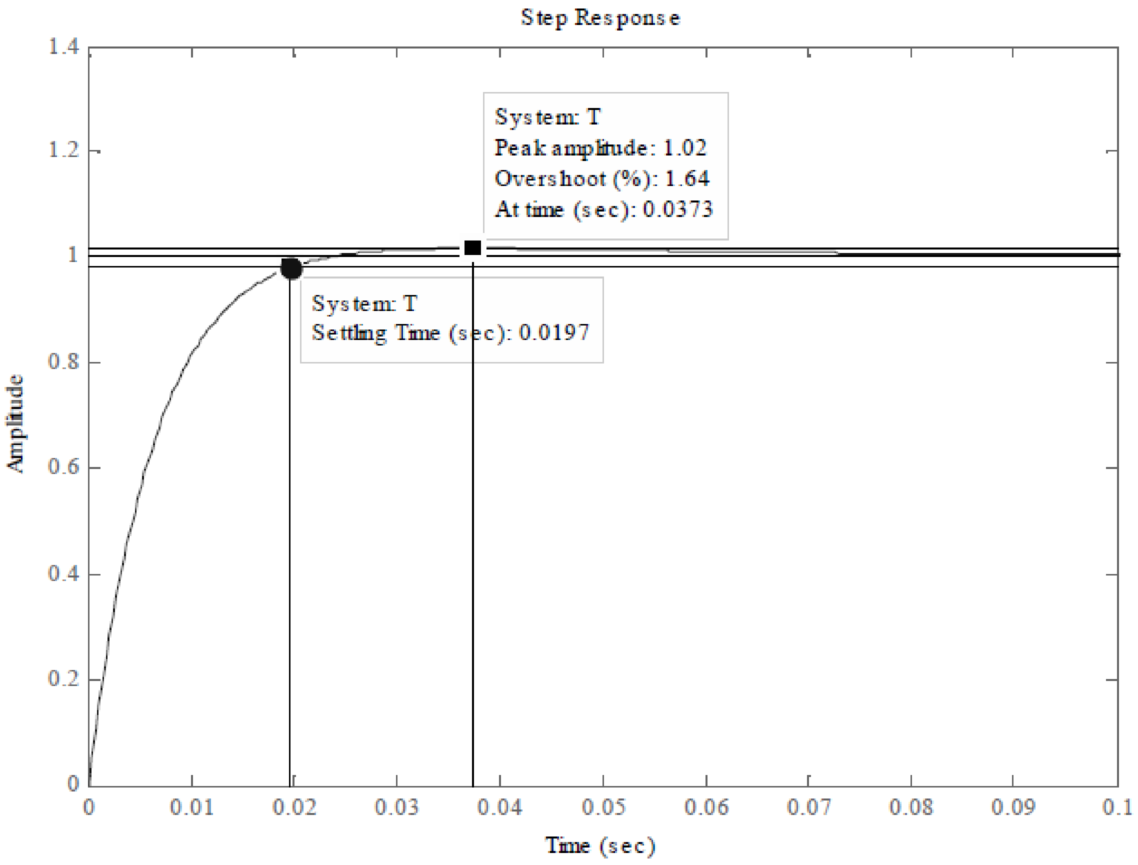
$$e_{ss}=0 \quad PO \leq 2\% \quad t_s \leq 0.08s$$



PO bigger than the design because the effect of zero.

Disk drive control system improvement – PD controller

$$e_{ss}=0 \quad PO\leq 2\% \qquad t_s \leq 0.08s$$



$$K_p = 822.45$$

$$K_d = 32$$

Tweak K_d value to achieve the design requirement.