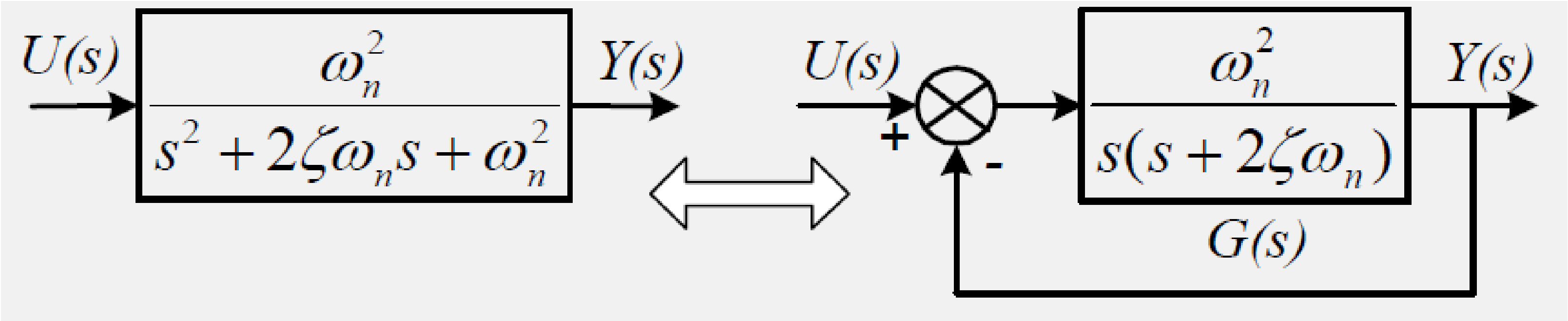
Instrumentation and Controls
ETM 3301

Lecture 25

Instructor

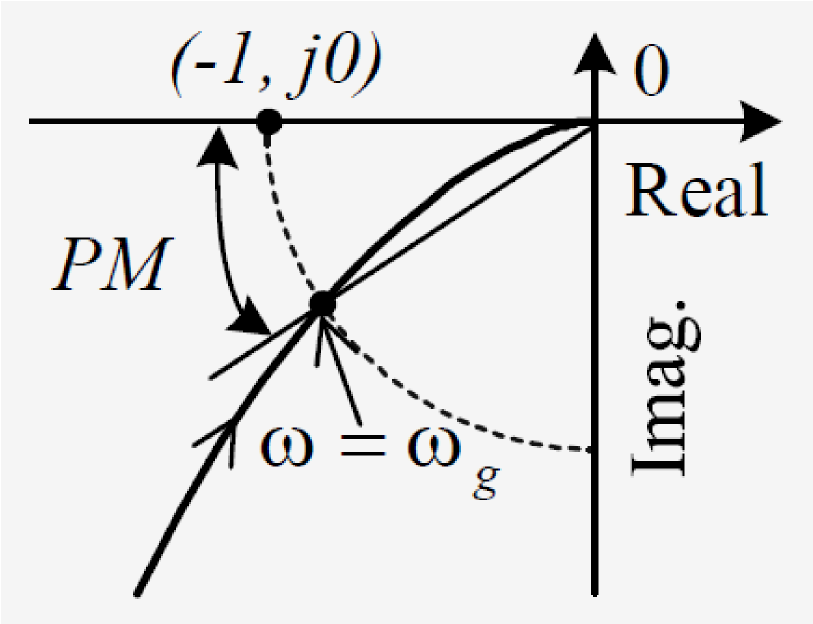
Dr. Farbod Khoshnoud

Phase Margin of Second Order Systems



- Open-loop frequency response:

$$G(j\omega)=\frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}$$



Gain Margin Calculation

- The ***gain margin*** is the increase in the system gain at phase ***-180º*** that will result in a marginally stable system with intersection of *(-1, j0)* point on the polar plot.

<u>*Step 1*</u>: Find the phase crossover frequency at which the phase is *-180º*.

$$\angle GH(j\omega_p) = -180^o$$

<u>*Step 2*</u>: Determine the gain at the phase crossover frequency and then calculate the gain margin.

$$\text{Gain Margin} \quad GM = \frac{1}{\left|GH(j\omega_p)\right|}$$

Phase Margin Calculation

- The phase margin is the angle by which the phase of $GH(j\omega)$ is short of -180^o when the gain is unity.

<u>*Step 1*</u>: Find the gain crossover frequency at which the gain is unity.

$$\left|GH(j\omega_g)\right| = 1$$

<u>*Step 2*</u>: Determine the phase at the gain crossover frequency and then calculate the phase margin.

$$\text{Phase margin } \; PM = \phi = 180^o + \angle GH(j\omega_g)$$

Phase Margin of Second Order Systems

$$|G(j\omega)| = \left|\frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}\right| = \frac{|\omega_n^2|}{|j\omega(j\omega+2\zeta\omega_n)|}$$

$$= \frac{|\omega_n^2|}{|j\omega||j\omega+2\zeta\omega_n|} = \frac{\omega_n^2}{\omega\sqrt{\omega^2+(2\zeta\omega_n)^2}}$$

Step 1: Find the gain crossover frequency where the gain is 1.

$$|G(j\omega_g)| = \frac{\omega_n^2}{\omega_g\sqrt{\omega_g^2+(2\zeta\omega_n)^2}} = 1$$

$$\frac{\omega_n^4}{\omega_g^2\left(\omega_g^2+(2\zeta\omega_n)^2\right)} = 1 \quad \Rightarrow \quad \omega_n^4 = \omega_g^2\left(\omega_g^2+(2\zeta\omega_n)^2\right)$$

Phase Margin of Second Order Systems

$$\omega_n^4 = \omega_g^2\left(\omega_g^2 + (2\zeta\omega_n)^2\right) \quad \text{assume} \quad x = \omega_g^2$$

$$\Rightarrow \omega_n^4 = x\left(x + (2\zeta\omega_n)^2\right) \quad \Rightarrow x^2 + 4\zeta^2\omega_n^2 x - \omega_n^4 = 0$$

$$\Rightarrow x = \frac{-4\zeta^2\omega_n^2 \pm \sqrt{(4\zeta^2\omega_n^2)^2 + 4\omega_n^4}}{2}$$

$$\Rightarrow x = -2\zeta^2\omega_n^2 \pm \sqrt{4\zeta^4 + 1}\,\omega_n^2 \quad (x \text{ must be positive})$$

$$\Rightarrow x = \left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)\omega_n^2$$

$$\omega_g = \sqrt{x} = \omega_n\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}$$

Phase Margin of Second Order Systems

$$\angle G(j\omega) = \angle\left(\frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}\right)$$

$$= \angle\omega_n^2 - \angle\left(j\omega(j\omega + 2\zeta\omega_n)\right)$$

$$= 0 - \left(\angle(j\omega) + \angle(j\omega + 2\zeta\omega_n)\right) = -90^o - \tan^{-1}\frac{\omega}{2\zeta\omega_n}$$

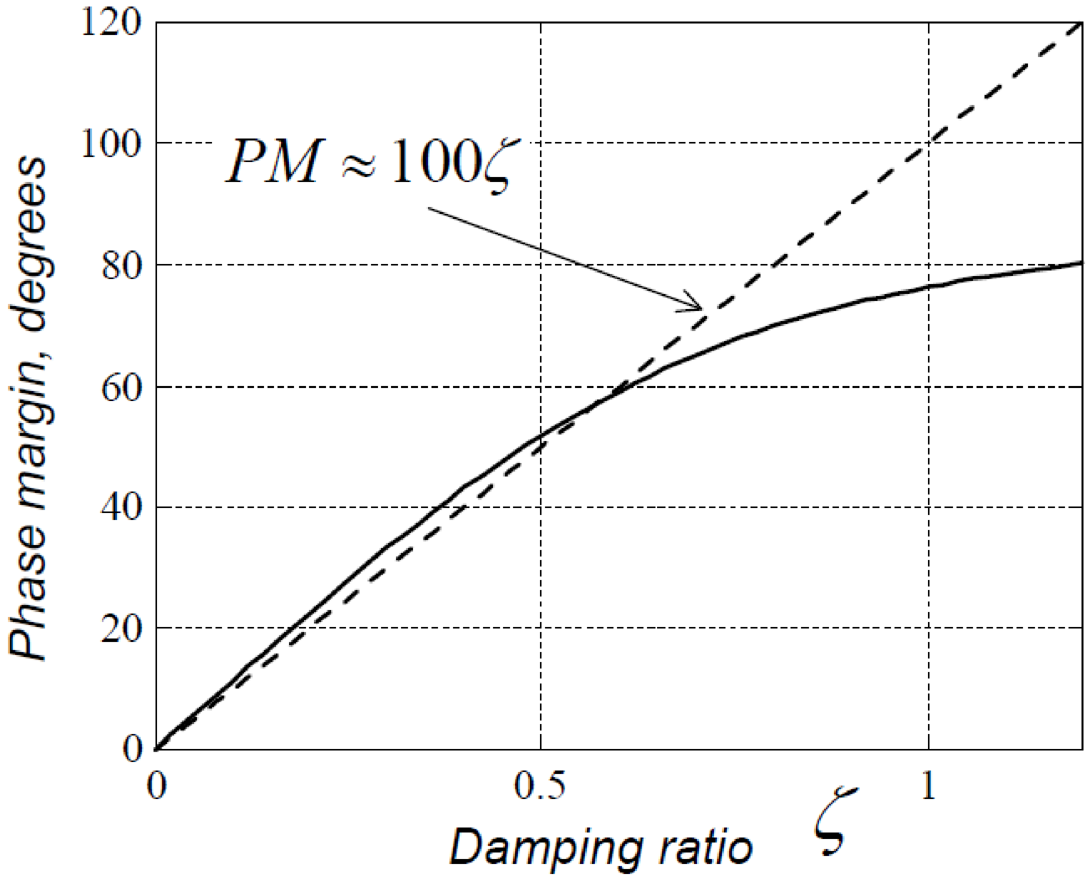
Step 2: Determine the phase at the gain crossover frequency and then calculate the phase margin.

$$\omega_g = \omega_n\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}$$

$$PM = 180° + \angle G(j\omega_g)$$

$$= \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}}$$

Phase Margin of Second Order Systems



Linear approximation is derived using 2nd order system, but this approximation is also applicable to higher order systems, 3rd, 4th, 5th etc

$$PM = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}}$$

Linear approximation

$PM \approx 100\zeta$

when $0<\zeta<0.7$

Phase Margin Approximation Example

Find the phase margin for a system with the OLTF:

$$G(s) = \frac{9}{s(s+3)} \xrightarrow{s=j\omega} G(j\omega) = \frac{9}{j\omega(j\omega+3)}$$

$$|G(j\omega)| = \left|\frac{9}{j\omega(j\omega+3)}\right| = \frac{|9|}{|j\omega(j\omega+3)|}$$

$$= \frac{9}{|j\omega||j\omega+3|} = \frac{9}{\omega\sqrt{9+\omega^2}}$$

We need to first find gain crossover frequency.

$$|G(j\omega_g)| = \frac{9}{\omega_g\sqrt{9+\omega_g^2}} = 1 \quad \Rightarrow \omega_g^4 + 9\omega_g^2 - 81 = 0$$

Phase Margin Approximation Example

$$\omega_g^4 + 9\omega_g^2 - 81 = 0$$

$$\Rightarrow \quad \omega_g^2 = \frac{-9 \pm \sqrt{9^2 + 4 \times 81}}{2} = \frac{-9 \pm 20.12}{2}$$

$$\Rightarrow \omega_g^2 = \frac{-9 + 20.12}{2} = 5.56 \quad \Rightarrow \quad \omega_g = 2.36$$

$$\angle G(j\omega) = \angle\left(\frac{9}{j\omega(j\omega+3)}\right) = \angle 9 - \angle\big(j\omega(j\omega+3)\big)$$

$$= 0 - \big(\angle(j\omega) + \angle(j\omega+3)\big) = -90^\circ - \tan^{-1}\frac{\omega}{3}$$

Phase Margin Approximation Example

$$\angle G(j\omega) = -90° - \tan^{-1}\frac{\omega}{3}$$

$$PM = 180° + \angle G(j\omega_g) = 180° - 90° - \tan^{-1}\frac{\omega_g}{3}$$

$$= 90° - 38.2° = 51.8°$$

Approximation method

$$G(s) = \frac{9}{s(s+3)} \Longleftrightarrow G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

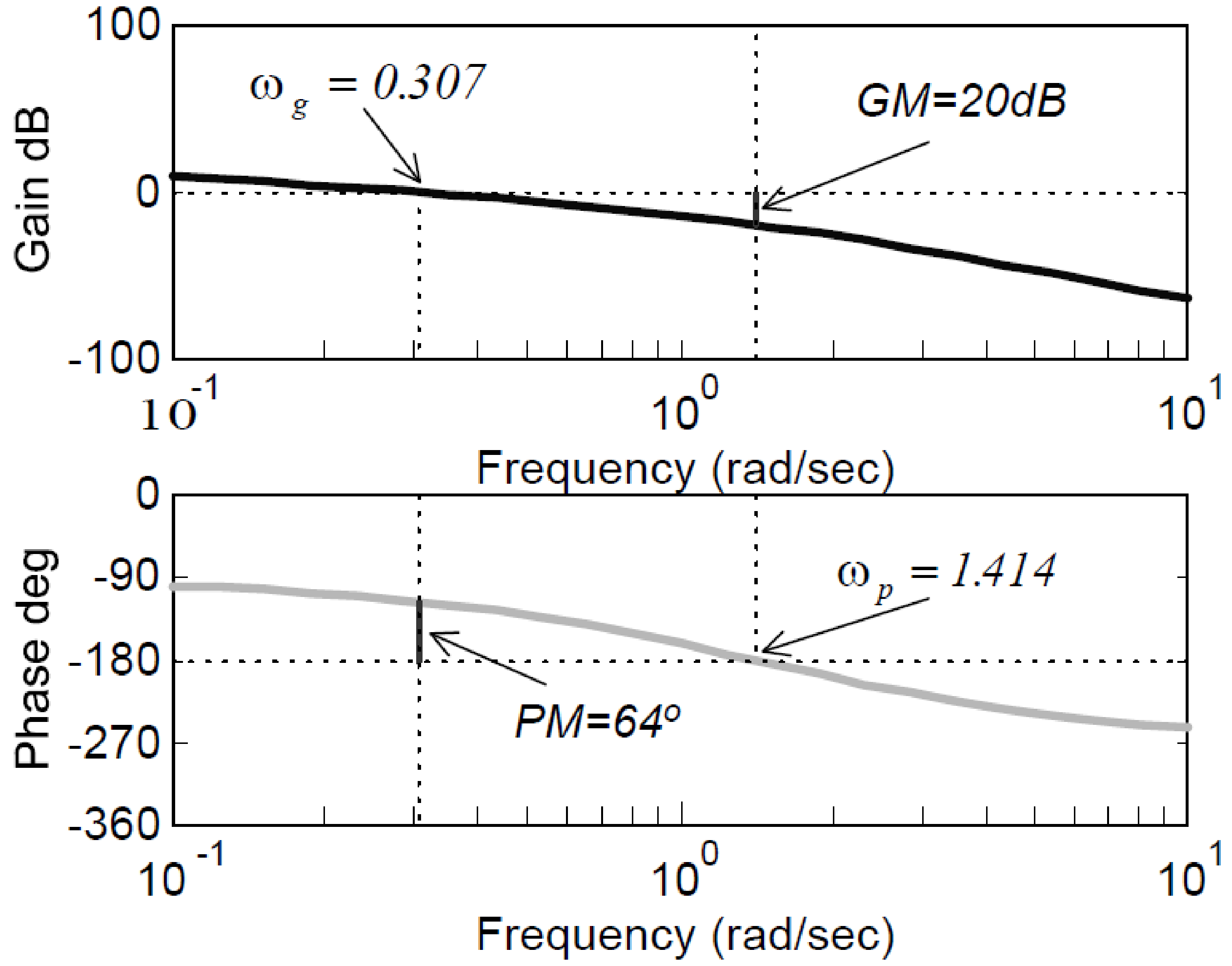
$$\omega_n^2 = 9 \Rightarrow \omega_n = \sqrt{9} = 3 \qquad 2\zeta\omega_n = 3 \Rightarrow \zeta = 0.5 < 0.7$$

$$PM \approx 100\zeta = 50°$$

Bode plot gain and phase margins example

Open-loop transfer function

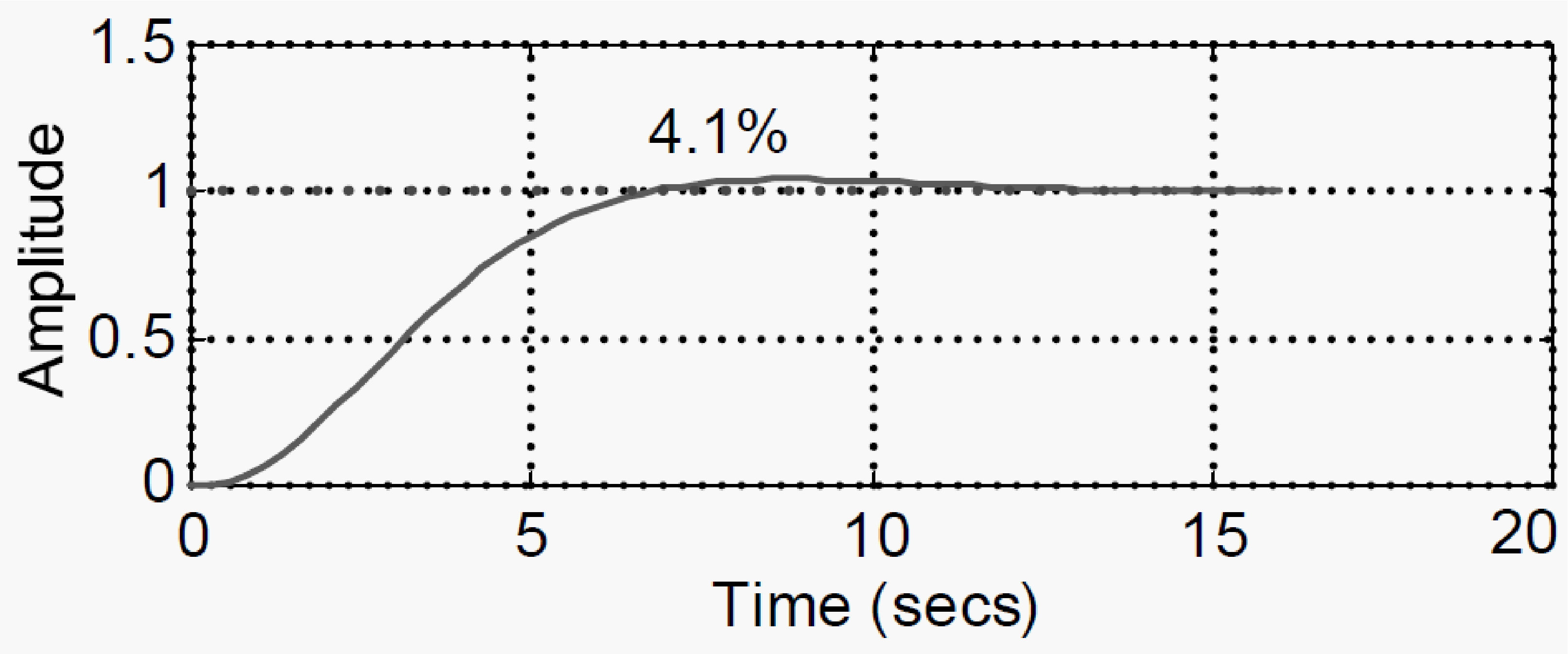
$$G(s) = \frac{0.325}{s(s+1)(0.5s+1)}$$



Bode plot gain and phase margins example

Equivalent damping ratio $\zeta \approx \frac{PM}{100} = 0.64$

Loop closed, predicted *PO* for step input $\approx 7\%$ ← $PO = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$



Control System Design using Frequency Response Method

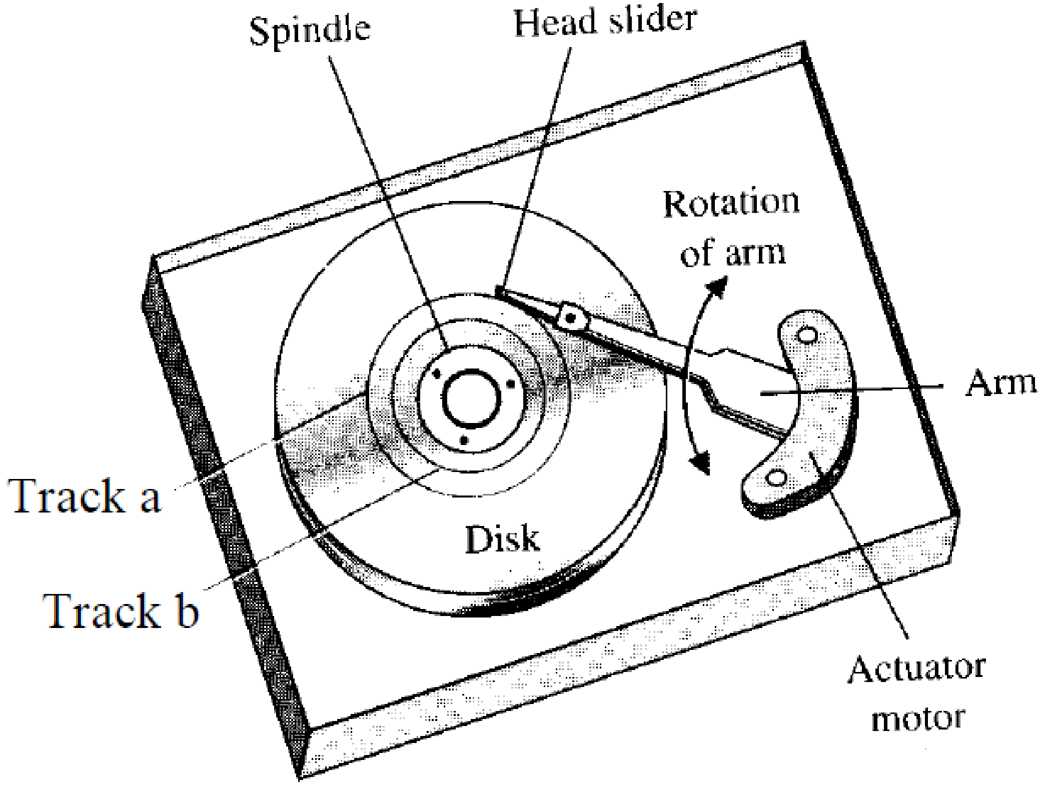
- Disk drive control system example
- Relationship between control system performance and frequency response
- Compensator for improving control performance
- Bode plot
- Phase-lead compensator design and implementation
- Frequency response design method using Matlab design tool

Disk drive control system example

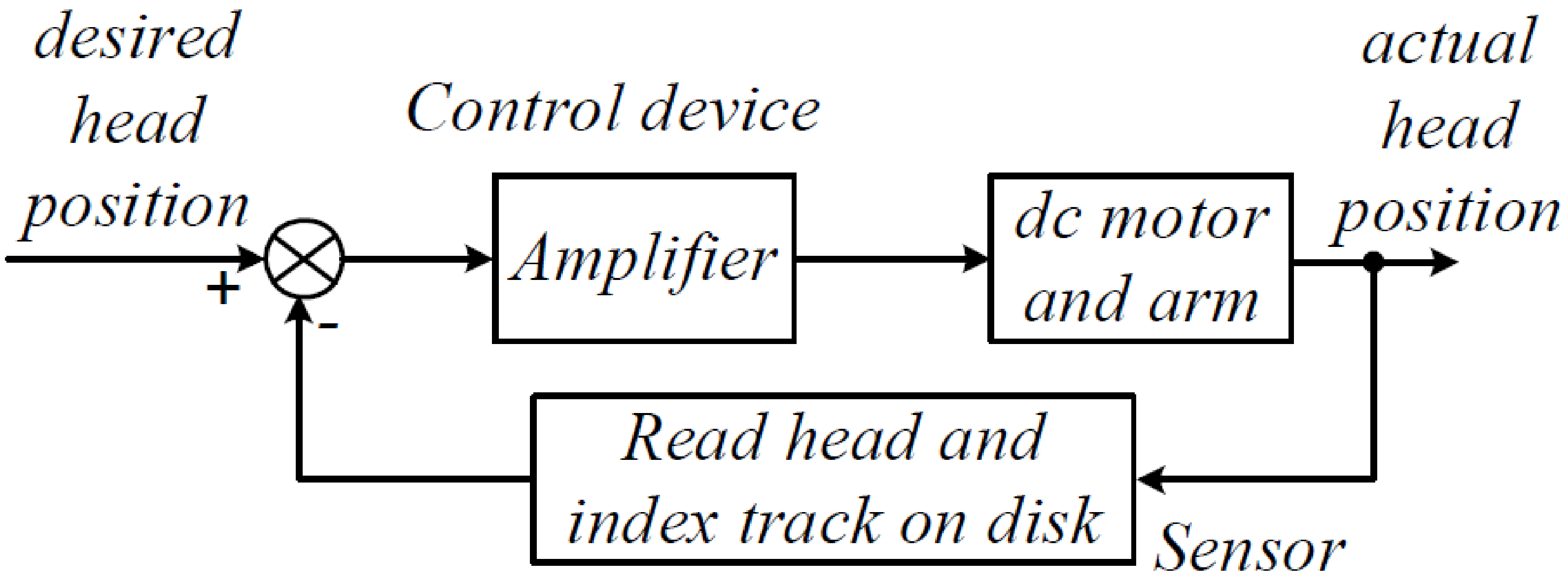
- Performance can be improved by introducing angular velocity feedback.

Design requirements: for a unit step input

$e_{ss}=0 \quad PO \leq 2\% \quad t_s \leq 0.08s$



Disk drive control system: schematic diagram

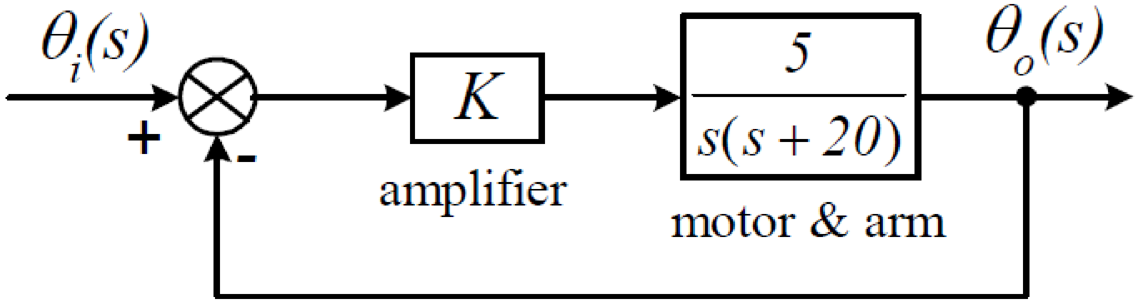


Modern hard drive: seek time *8.9ms*

When we put all components together, how to check the required performance is achieved? If not, how to improve it?

PO should be small, less vibration, less noise.

Disk drive control system Design



K in the range of *10* to *1000* (Physical Restriction)

Settling time: *0.4* seconds when *K=40*

TF: $$T(s) = \frac{5K}{s^2 + 20s + 5K} \qquad \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\zeta\omega_n = 10$$

$$\left.\begin{aligned} 2\zeta\omega_n &= 20 \\ \omega_n^2 &= 5K \end{aligned}\right\} \Rightarrow \omega_n = \sqrt{5K}; \quad \zeta = \frac{10}{\sqrt{5K}}$$

Disk drive control system design

$$\zeta\omega_n = 10 \quad \omega_n = \sqrt{5K} \quad \zeta = \frac{10}{\sqrt{5K}}$$

Underdamped design: $0 < \zeta < 1 \Rightarrow K > 20$

$$t_s = \frac{4}{\zeta\omega_n} = 0.4\,\text{sec}$$

Independent of K!! We cannot improve settling time by changing K alone.

$$\zeta = \frac{10}{\sqrt{5K}}; \quad PO = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

Depends on K.

K	30	40	100
ζ	0.82	0.71	0.45
PO	1.2%	4.3%	20.8%

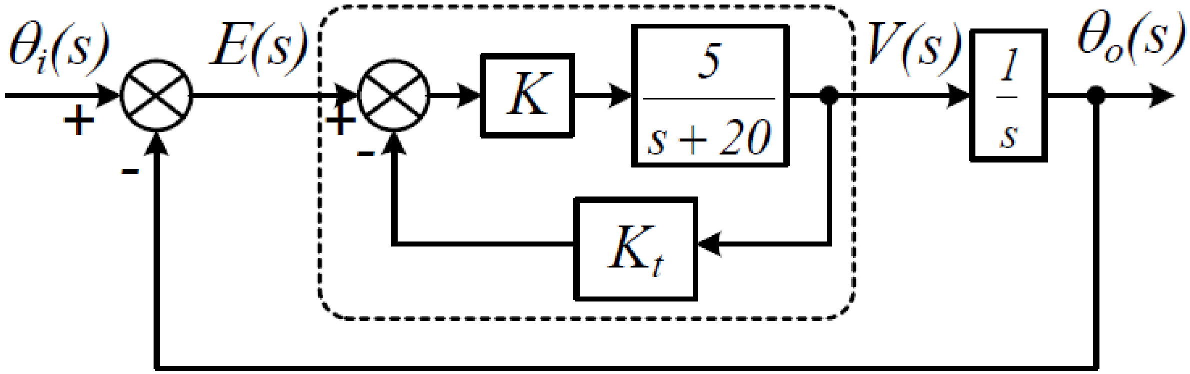
$t_s = 0.4\,\text{sec}$

Disk drive control system improvement

- Performance can be improved by introducing angular velocity feedback.

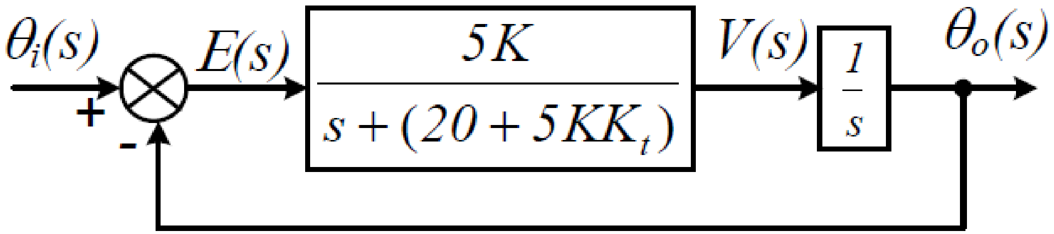
Design requirements: for a unit step input

$e_{ss}=0 \quad PO\leq 2\% \quad t_s \leq 0.08s$



$$\frac{V(s)}{E(s)} = \frac{\frac{5K}{s+20}}{1+\frac{5K}{s+20}K_t} = \frac{5K}{s+(20+5KK_t)}$$

Disk drive control system improvement



Forward path TF: $$G(s) = \frac{5K}{s(s + (20 + 5KK_t))}$$

The overall closed-loop transfer function:

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{G(s)}{1 + G(s)} = \frac{5K}{s^2 + (20 + 5KK_t)s + 5K}$$

Steady-state error for a unit step input:

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0 \qquad K_p = \lim_{s \to 0} G(s) = \infty$$

Disk drive control system improvement

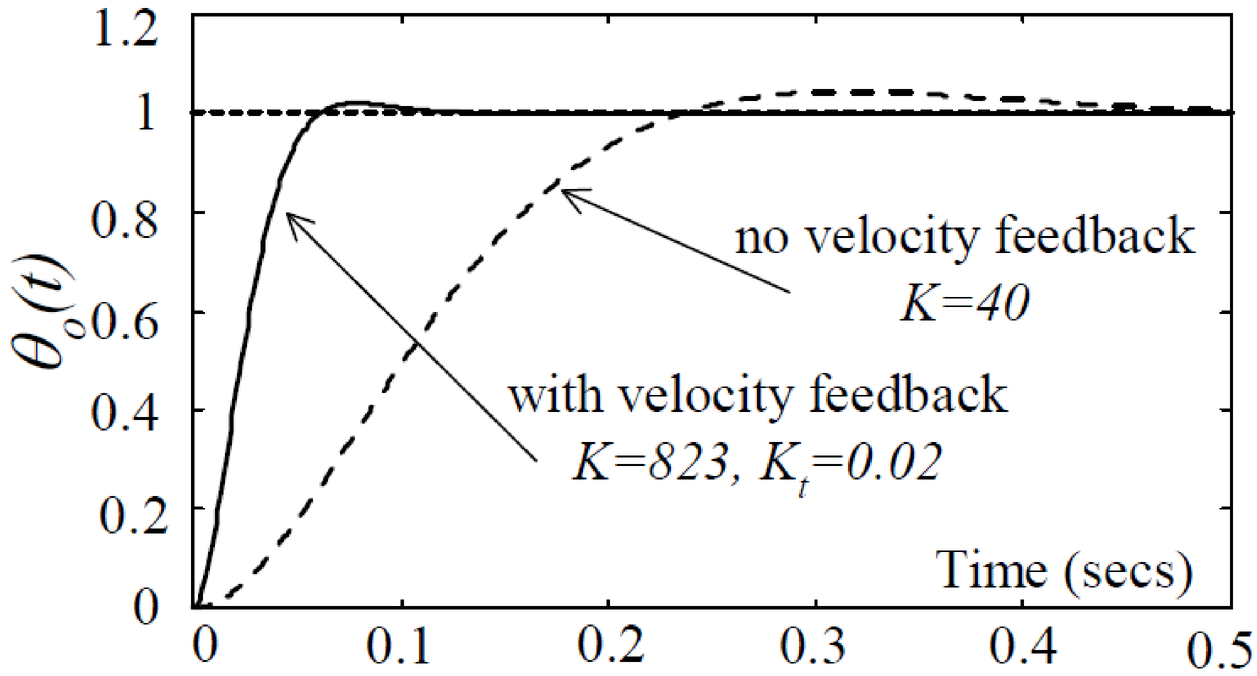
TF: $$T(s) = \frac{5K}{s^2 + (20 + 5KK_t)s + 5K} \qquad \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\left.\begin{aligned} 2\zeta\omega_n &= 20 + 5KK_t \\ \omega_n^2 &= 5K \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \omega_n &= \sqrt{5K} \\ \zeta &= \frac{20 + 5KK_t}{2\sqrt{5K}} \end{aligned}\right.$$

K	K_t	ζ	ω_n	PO	t_s
400	0.0216	0.707	44.721	4.3%	0.1265
823	0.02	0.78	64.1	2%	0.08
960	0.0204	0.85	69.282	0.63%	0.0679
46451	0.0034	0.84	481.93	0.77%	0.0099

Disk drive control system improvement

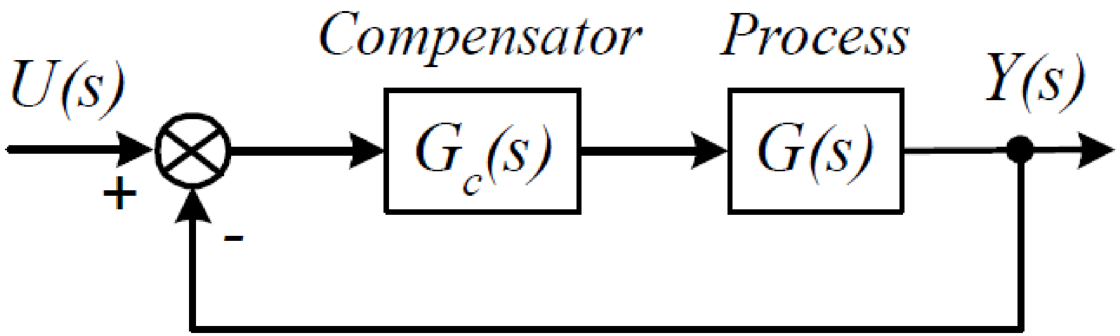
- Performance can be improved by introducing angular velocity feedback.

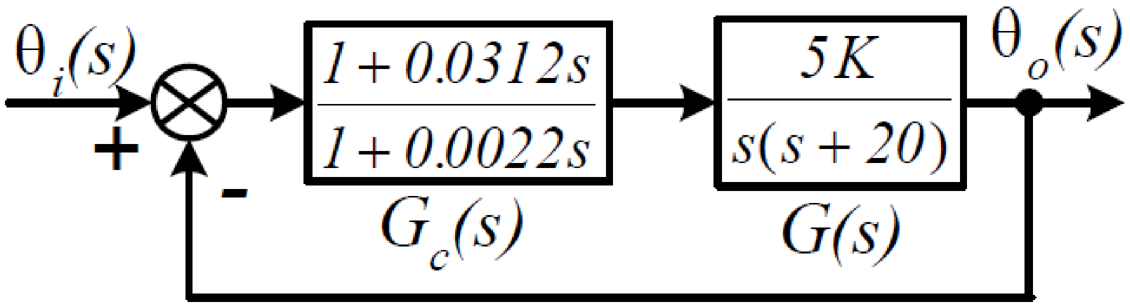


$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{4115}{s^2 + 100s + 4115}$$

Disk drive control system improvement

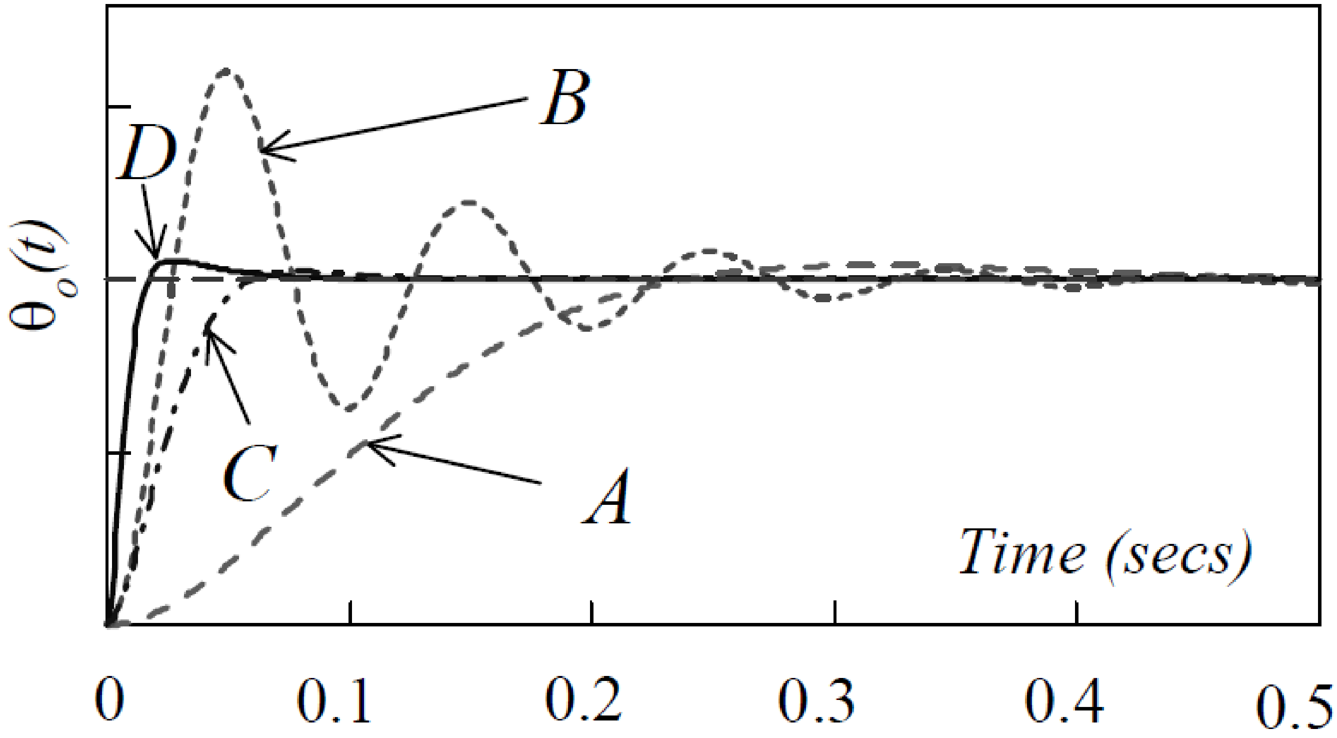
- An additional sensor (e.g. tachometer) which measures angular velocity is required to implement velocity feedback.
- A cheap alternative: use a compensator.





A compensator (controller) is inserted into the forward path of the system in order to improve control performance.

Disk drive control system improvement



A: no compensator, no velocity feedback, $K=40$

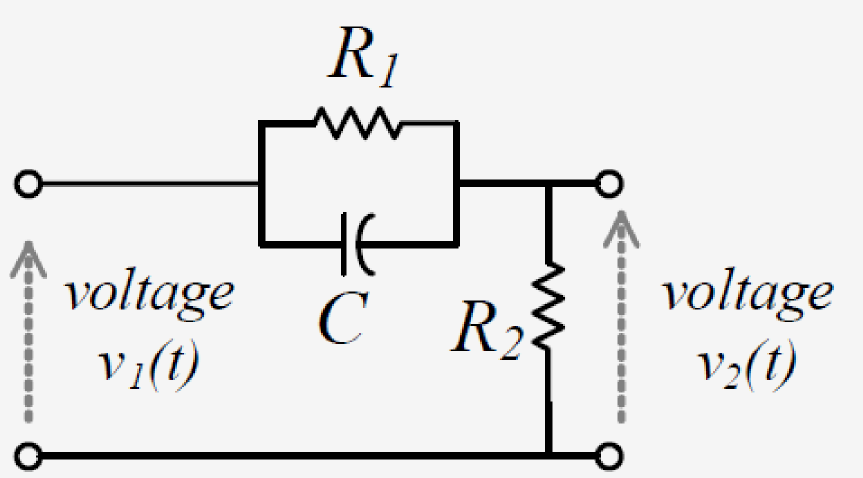
B: no compensator, no velocity feedback, $K=823$

C: no compensator, velocity feedback $K=823$, $K_t=0.02$

D with compensator, no velocity feedback, $K=823$

Disk drive control system improvement

- Physical implementation of compensator.



$$G_N(s) = \frac{V_2}{V_1} = \frac{R_2}{R_2 + \frac{R_1 \frac{1}{CS}}{R_1 + \frac{1}{CS}}}$$

$$= \frac{R_2}{R_1 + R_2} \frac{(1 + R_1 Cs)}{1 + \frac{R_1 R_2 C}{R_1 + R_2} s}$$

Resistor:

$$v(t) = Ri(t)$$

$$\Rightarrow V(s) = \boxed{R} \; I(s)$$

Capacitor:

$$i(t) = C\frac{dv(t)}{dt}$$

$$\Rightarrow I(s) = CsV(s)$$

$$\Rightarrow V(s) = \boxed{\frac{1}{Cs}} I(s)$$

Ohm's Law

Disk drive control system improvement, 9

$$G_N(s) = \frac{R_2}{R_1 + R_2} \frac{(1 + R_1 C s)}{1 + \frac{R_1 R_2 C}{R_1 + R_2} s} \qquad \alpha \equiv \frac{R_1 + R_2}{R_2} \qquad T \equiv \frac{R_1 R_2 C}{R_1 + R_2}$$

$$G_N(s) = \left(\frac{1}{\alpha}\right)\left(\frac{1 + \alpha T s}{1 + T s}\right)$$

$$R_1 = 395\Omega,\ R_2 = 30\Omega,\ C = 78.9\mu F$$

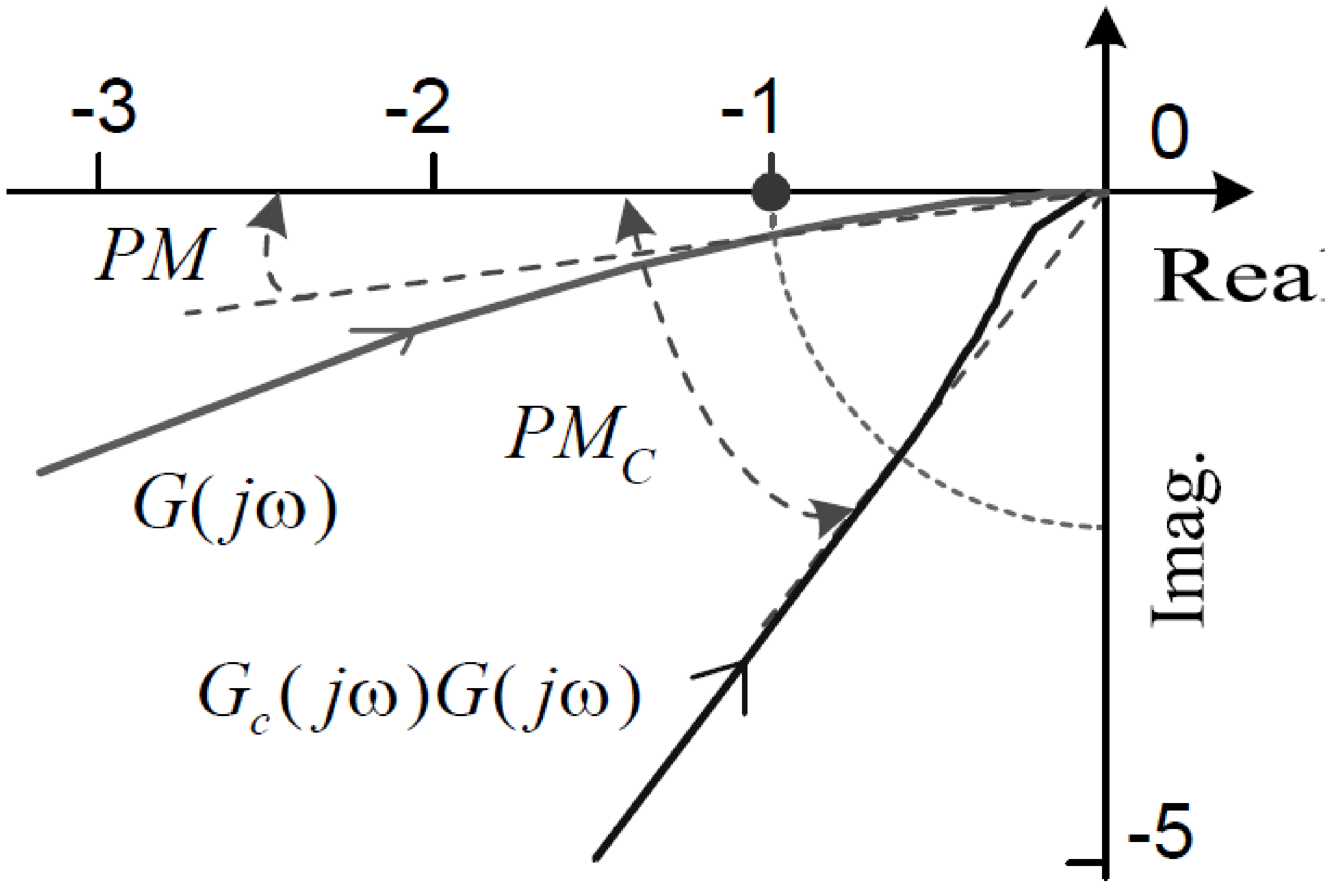
$$G_N(s) = \left(\frac{1}{14.2}\right)\left(\frac{1 + 0.0312 s}{1 + 0.0022 s}\right)$$

The RC network connected to a amplifier of gain *14.2* will give us the required compensator transfer function.

Disk drive control, compensator effect

	uncompensated		compensated	
ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	$\lvert GG_c(j\omega)\rvert$	$\angle GG_c(j\omega)$
30	3.8	-146.3°	5.2	-107.0°
40	2.3	-153.4°	3.7	-107.2°
50	1.5	-158.2°	2.8	-107.1°
62.6	1.0	-162.3°	2.2	-107.2°
80	0.6	-166.0°	1.7	-107.8°
100	0.4	-168.7°	1.3	-108.9°
126.1	0.26	-171.0°	1.0	-110.8°
150	0.18	-172.4°	0.9	-112.7°
300	0.05	-176.2°	0.36	-125.7°
1000	0.0	-178.9°	0.05	-156.2°

Disk drive control, compensator effect



Phase margin: $17.7° \rightarrow 69.2°$

Link Between Control System Transient Performance and Frequency Response

- Transient performance: percentage overshoot (PO) and settling time (t_s)

$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

- Frequency response: gain margin (GM) and phase margin (PM)

$$PM \approx 100\zeta \quad \text{when} \quad 0<\zeta<0.7$$

By increasing phase margin, the percentage overshoot can be reduced.