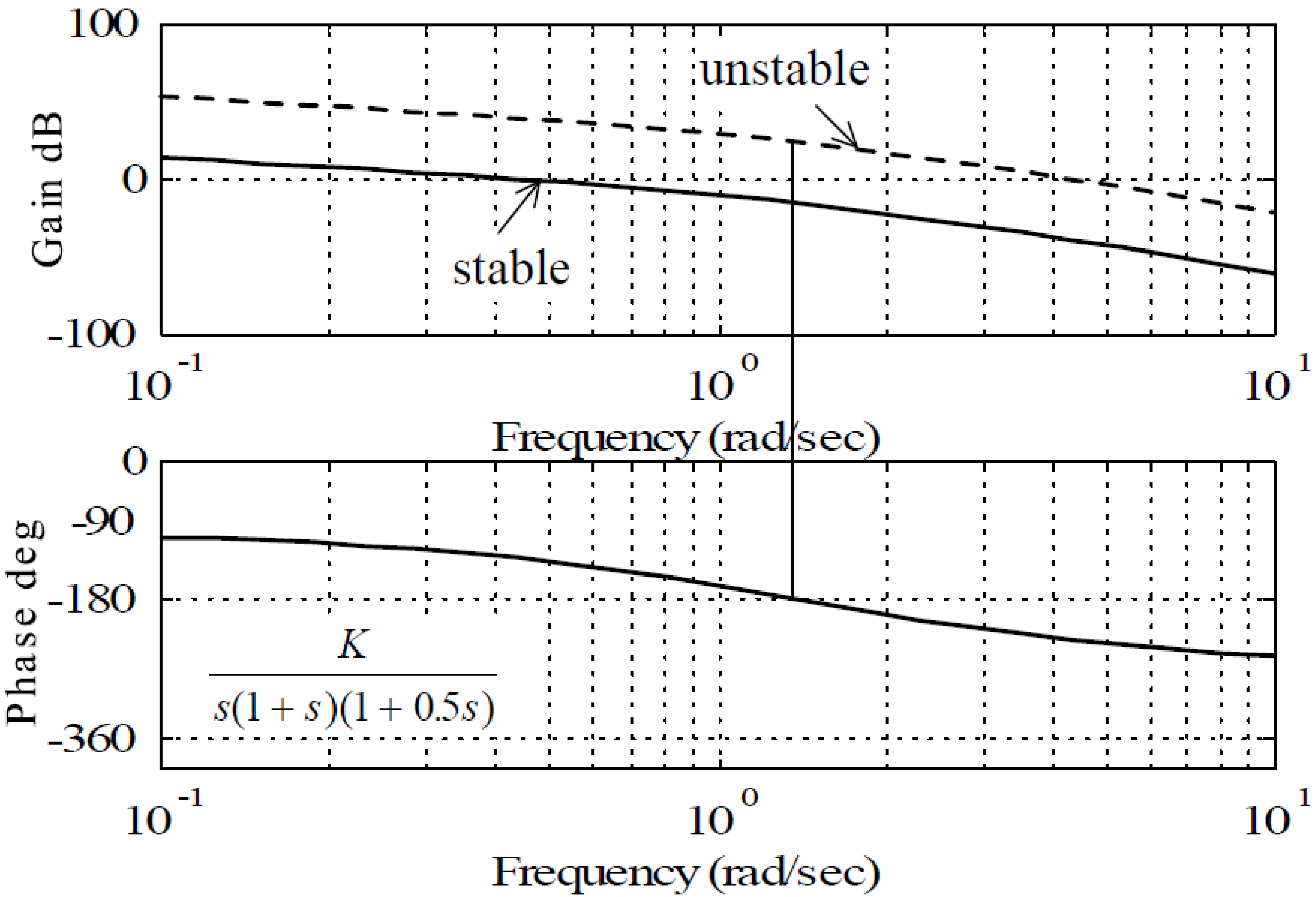
Instrumentation and Controls
ETM 3301

Lecture 24

Instructor
Dr. Farbod Khoshnoud

Nyquist stability criterion using Bode plot

- The CL system is stable if the gain is less than *0dB* when the phase is -180^o .



Check Closed-Loop Stability for Open-Loop Unstable Systems

- P: the number of open-loop poles with positive real parts
- Z: the number of closed-loop poles with positive real parts
- N: the number of <u>*clockwise encirclements*</u> of the point $(-1, j0)$ made by $GH(j\omega)$ when ω varies from $-\infty$ to $+\infty$.

$$Z = P + N$$

$Z = 0$ stable

Closed-Loop Stability for Open-Loop Unstable Systems

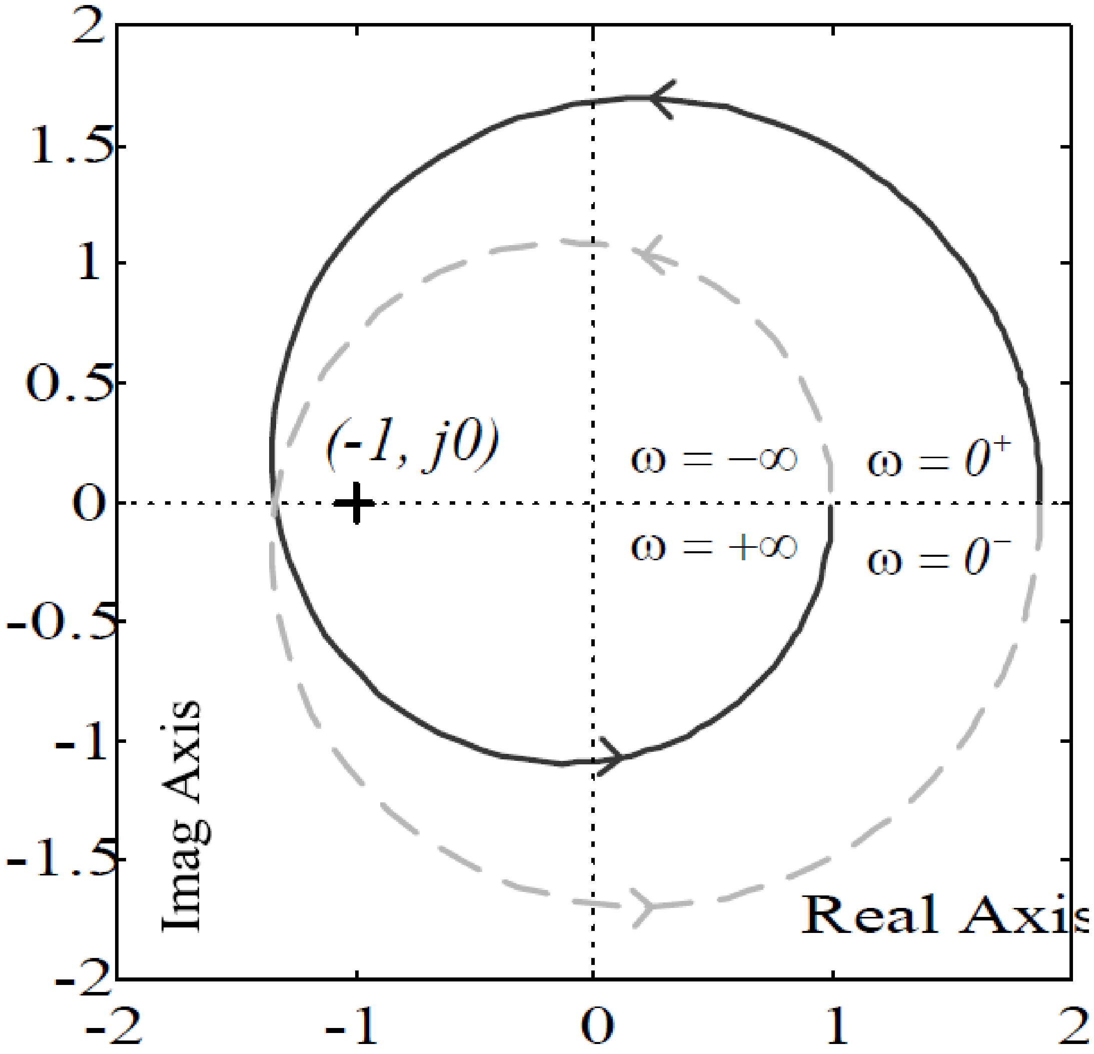
$$GH(s)=\frac{(s+3)(s+5)}{(s-2)(s-4)}$$

$P=2$

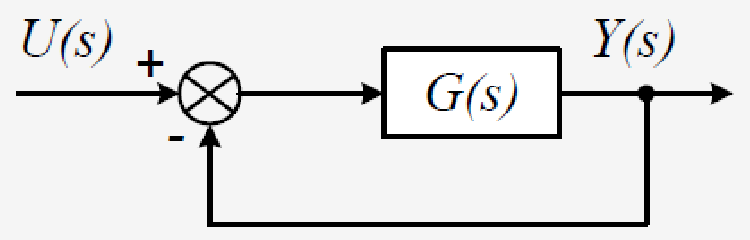
$N=-2$

$Z=P+N=0$

stable



Stability Margin Concept

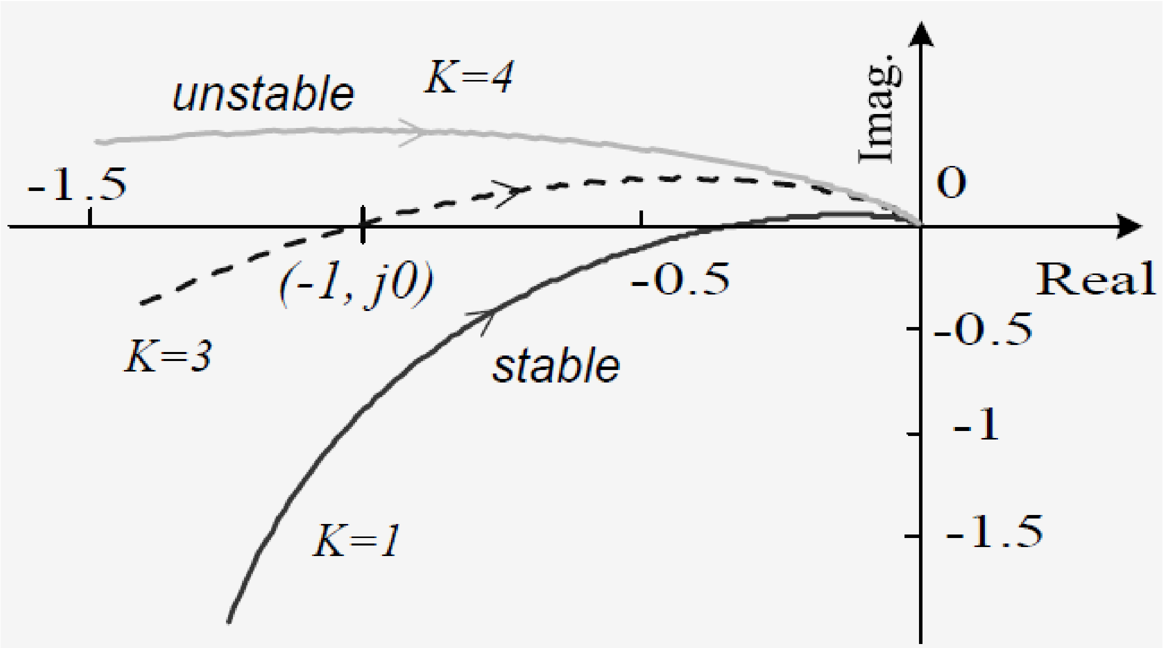


Open-loop TF:

$$G(s) = \frac{K}{s(s+1)(0.5s+1)}$$

Poles: *0, -1, -2* Stable

The system is *conditionally* (or *marginally*) stable.



- *K=1* the closed-loop system is stable
  - the CL system may become unstable when the *K* is increased.
  - the safety margin for stability is 3/1=3

Stability Margin

- For a system with the stable open loop transfer function (OLTF), the closed-loop system may be stable and may also be unstable.
  - It is useful to know to *what extent* the OLTF can be modified before instability occurs.
  - The ***gain margin*** and ***phase margin*** provide this type of information.
  - i.e how stable the system is.

Relative Stability

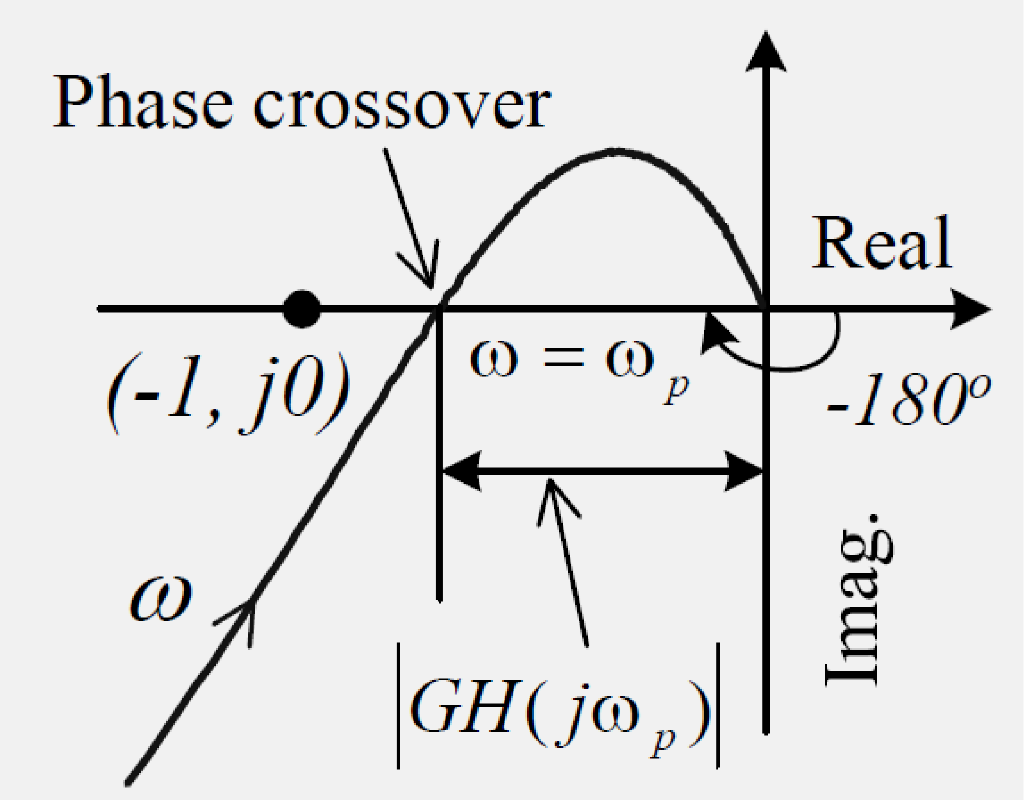
Defining Stability Margin

- According to Nyquist Stability Criterion, the point *(-1, j0)* in the polar plot is the critical point for the stability.
  - If the polar plot curve passes through *(-1, j0)*, the system is on the verge of instability.
  - Gain margin and phase margin are measures which specify the distance of the polar plot to *(-1, j0)*.

Point *(-1, j0)*	
Gain: *1*	Phase: *-180°*

Gain Margin Definition

- The ***gain margin*** is the amount by which the gain can be increased before the system becomes unstable.



Gain Margin

$$GM = \frac{1}{|GH(j\omega_p)|}$$

Stable Systems

$$GM > 1$$

Gain Margin Calculation

- The ***gain margin*** is the increase in the system gain at phase ***-180º*** that will result in a marginally stable system with intersection of *(-1, j0)* point on the polar plot.

<u>*Step 1*</u>: Find the phase crossover frequency at which the phase is *-180º*.

$$\angle GH(j\omega_p) = -180^o$$

<u>*Step 2*</u>: Determine the gain at the phase crossover frequency and then calculate the gain margin.

$$\text{Gain Margin} \quad GM = \frac{1}{|GH(j\omega_p)|}$$

Gain Margin Example

- Find the gain margin for a system with the OL frequency response as:

$$G(j\omega) = \frac{K}{j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)}$$

<u>*Step 1*</u>: Find the phase crossover frequency where the phase is *-180°*.

$$\angle G(j\omega) = \angle\left(\frac{K}{j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)}\right)$$

$$= \angle K - \angle\left(j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)\right)$$

$$= \angle K - \left(\angle(j\omega) + \angle(T_1 j\omega + 1) + \angle(T_2 j\omega + 1)\right)$$

$$= -90^o - \tan^{-1} T_1\omega - \tan^{-1} T_2\omega$$

Gain Margin Example

<u>*Step 1*</u>: Find the phase crossover frequency where the phase is *-180°*.

$$\angle G(j\omega_p) = -90^o - \tan^{-1} T_1\omega_p - \tan^{-1} T_2\omega_p = -180^\circ$$

$$\Rightarrow \tan^{-1} T_1\omega_p + \tan^{-1} T_2\omega_p = 90^\circ$$

$$\Rightarrow \tan^{-1} \frac{T_1\omega_p + T_2\omega_p}{1 - T_1\omega_p \cdot T_2\omega_p} = 90^o$$

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1} \frac{A+B}{1-AB}$$

$$\Rightarrow \frac{T_1\omega_p + T_2\omega_p}{1 - T_1\omega_p \cdot T_2\omega_p} = \tan 90^o = \infty$$

$$\Rightarrow 1 - T_1\omega_p \cdot T_2\omega_p = 0$$

$$\omega_p = \frac{1}{\sqrt{T_1 T_2}}$$

Gain Margin Example

$$|G(j\omega)| = \left| \frac{K}{j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)} \right|$$

$$= \frac{|K|}{|j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)|}$$

$$= \frac{K}{|j\omega||T_1 j\omega + 1||T_2 j\omega + 1|}$$

$$= \frac{K}{\omega\sqrt{1 + (T_1\omega)^2}\sqrt{1 + (T_2\omega)^2}}$$

Gain Margin Example

Step 2: Determine the gain at the phase crossover frequency and then calculate the gain margin.

$$\left|G(j\omega_p)\right| = \frac{K}{\omega_p\sqrt{1+(T_1\omega_p)^2}\sqrt{1+(T_2\omega_p)^2}} \qquad \omega_p = \frac{1}{\sqrt{T_1T_2}}$$

$$\left|G(j\omega_p)\right| = \frac{K}{\frac{1}{\sqrt{T_1T_2}}\sqrt{1+T_1^2\frac{1}{T_1T_2}}\sqrt{1+T_2^2\frac{1}{T_1T_2}}}$$

$$= \frac{K}{\sqrt{\left(\frac{1}{T_1T_2}\right)\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)}} = \frac{KT_1T_2}{T_1+T_2}$$

Gain Margin Example

Find the gain margin:

$$GM = \frac{1}{|G(j\omega_p)|} = \frac{T_1 + T_2}{KT_1T_2}$$

For stable systems, $GM > 1$.

Hence the system stable when

$$\frac{T_1 + T_2}{KT_1T_2} > 1 \quad or \quad K < \frac{T_1 + T_2}{T_1T_2}$$

Gain Margin Example

Alternative method to find the phase crossover frequency:

$$G(j\omega) = \frac{K}{j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)}$$

$$= \frac{-jK(1 - T_1 j\omega)(1 - T_2 j\omega)}{\omega(1 + T_1^2\omega^2)(1 + T_2^2\omega^2)}$$

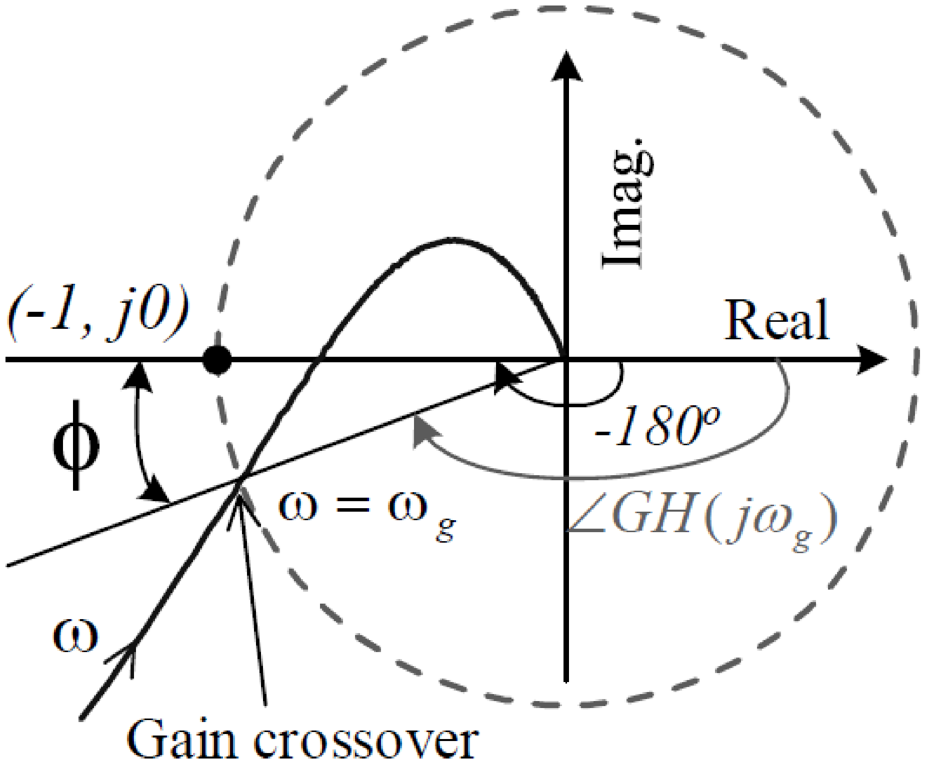
$$= \frac{-K(T_1 + T_2)\omega - jK(1 - T_1 T_2\omega^2)}{\omega(1 + T_1^2\omega^2)(1 + T_2^2\omega^2)}$$

$$\angle G(j\omega_p) = -180° \quad \Rightarrow \quad \mathrm{Im}(G(j\omega_p)) = 0$$

$$1 - T_1 T_2 \omega_p^2 = 0 \qquad \omega_p = \frac{1}{\sqrt{T_1 T_2}}$$

Phase Margin Definition

- The ***phase margin*** is the amount of phase shift of *GH(jω)* at unity gain that will result in a marginally stable system with intersection of *(-1, j0)* point on the polar plot.



Phase Margin ϕ

$$PM = 180° + \angle GH(j\omega_g)$$

Stable Systems

$$PM > 0$$

Phase Margin Calculation

- The phase margin is the angle by which the phase of $GH(j\omega)$ is short of -180^o when the gain is unity.

Step 1: Find the gain crossover frequency at which the gain is unity.

$$\left|GH(j\omega_g)\right| = 1$$

Step 2: Determine the phase at the gain crossover frequency and then calculate the phase margin.

$$\text{Phase margin } PM = \phi = 180^o + \angle GH(j\omega_g)$$

Phase Margin Example

Open loop frequency response:

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0.1	402.1	-174.7°
1	5.7	-140.1°
2	2.24	-127.6°
3.92	0.9998	-125.9°
7	0.481	-133.6°
10	0.289	-141.3°
15	0.1494	-150.7°
30	0.0423	-163.9°
40	0.0243	-167.7°

Gain crossover frequency:

$$\lvert GH(j\omega_g)\rvert = 1$$

$$\omega_g \approx 3.92$$

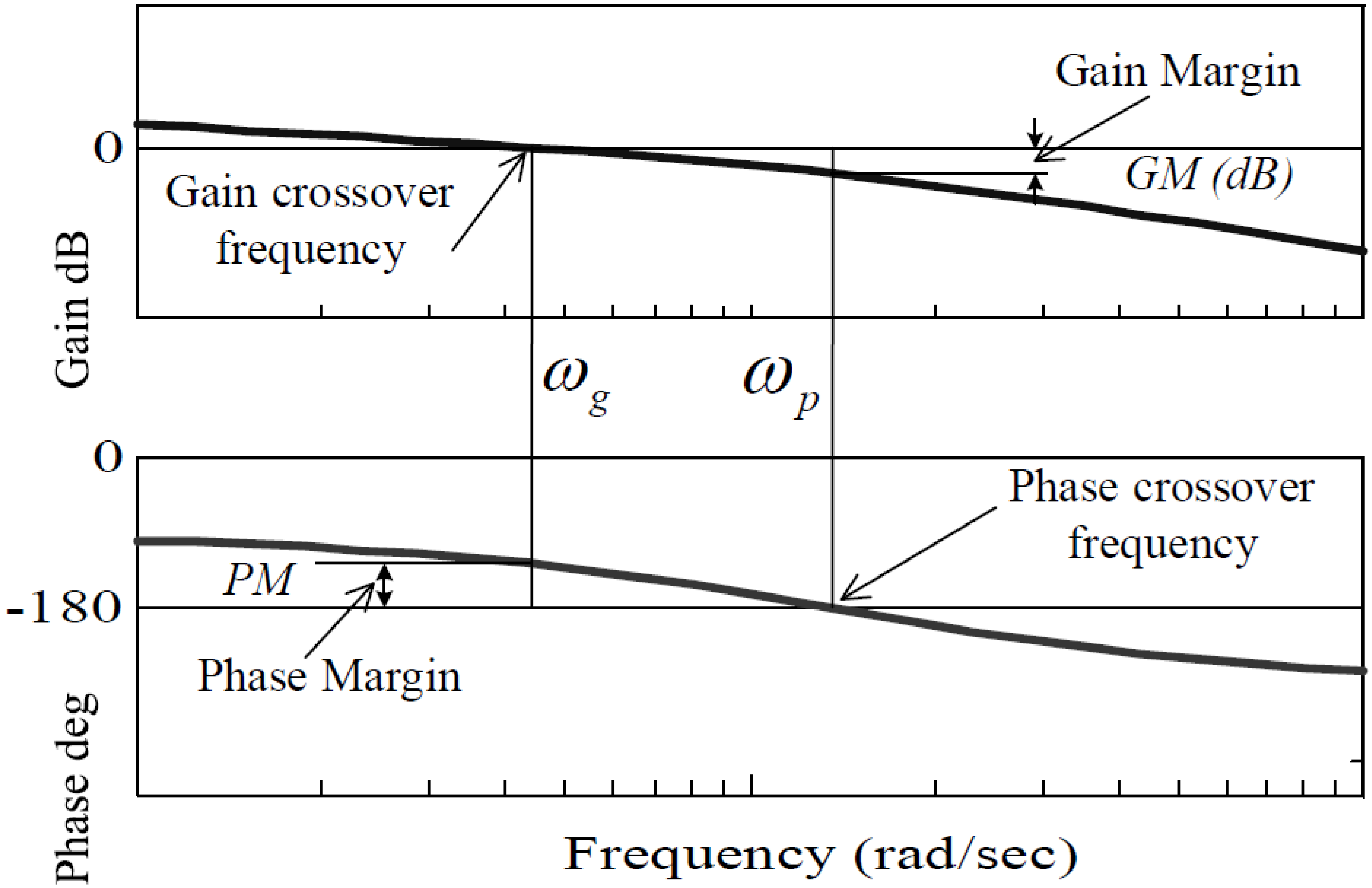
Phase margin:

$$PM = 180^o + \angle GH(j\omega_g)$$

$$\approx 180^o + (-125.9^o)$$

$$= 54.1^o$$

Determine gain and phase margins using Bode plot



Determine gain and phase margins using Bode plot

- ***Gain crossover frequency ω_g***: where the gain plot crosses the *0dB* axis.
- ***Phase margin (PM):*** the phase-angle curve above $-180°$ at the gain crossover frequency ω_g.
- ***Phase crossover frequency ω_p***: where the phase angle is $-180°$.
- ***Gain margin (GM):*** the distance of the gain below *0dB* at the phase crossover frequency ω_p.

Gain and Phase Margin Using Matlab

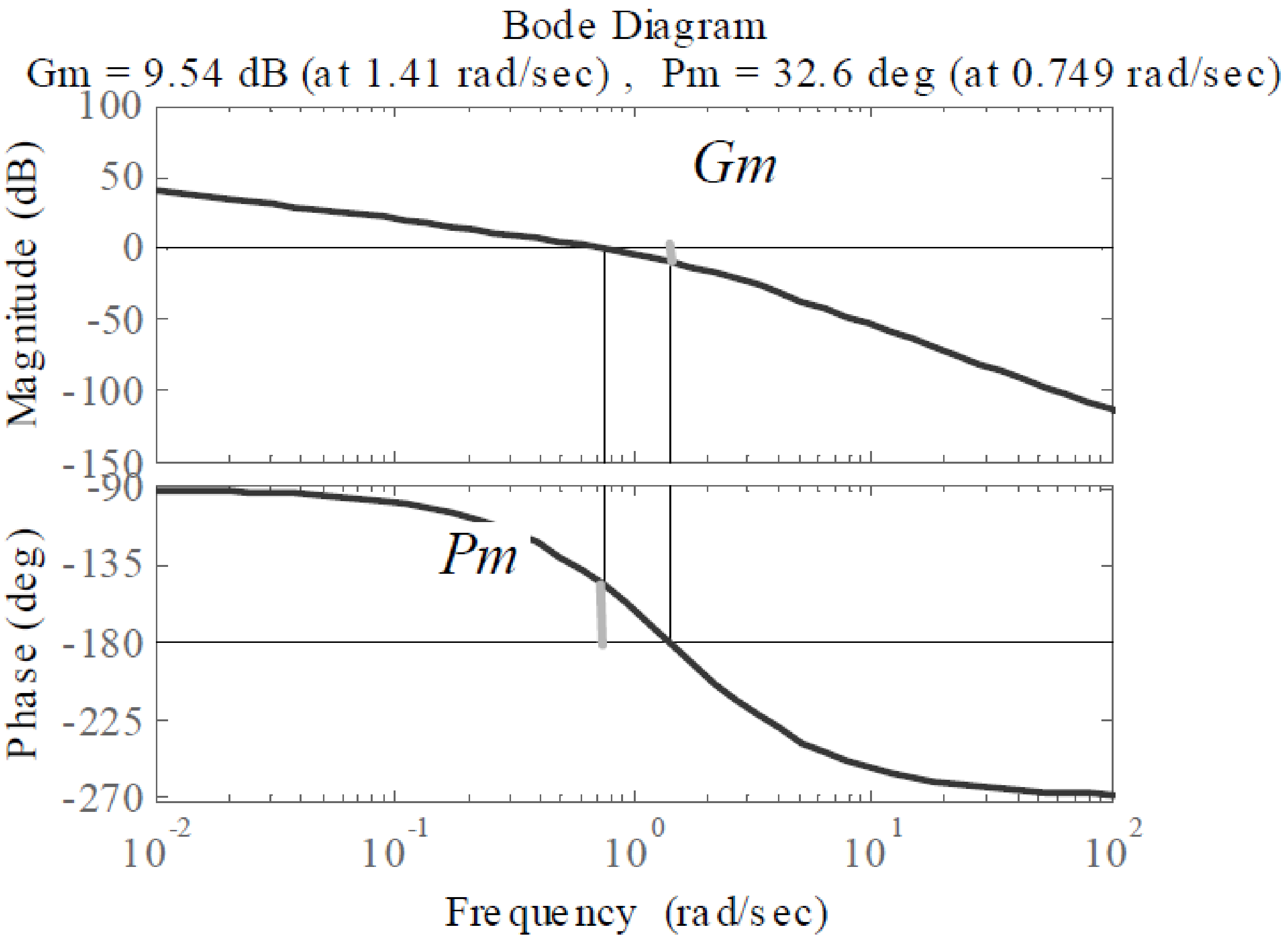
$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

```
>>G=tf([1],[0.5 1.5 1 0])
```

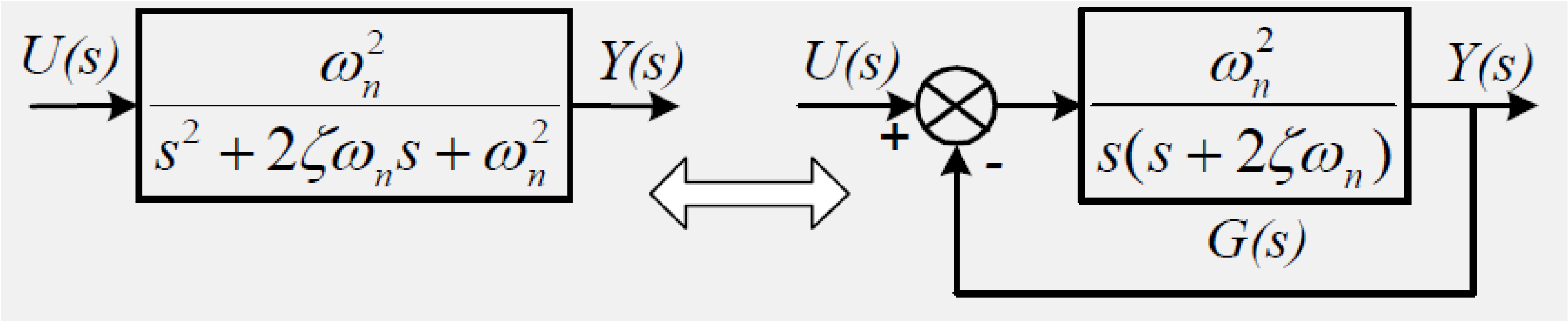
Or

```
>>G=zpk([],[0 -1 -2],[1])
```

```
>>margin(G)
```



Phase Margin of Second Order Systems



- Open-loop frequency response:

$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}$$