Instrumentation and Controls
ETM 3301

Lecture 23

Instructor
Dr. Farbod Khoshnoud

Nyquist Stability Criterion, Gain and Phase Stability Margins

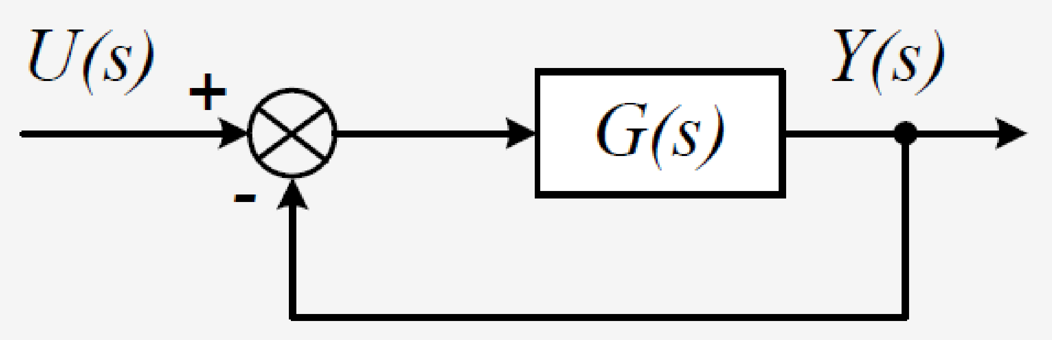
- Nyquist stability criterion.
- Applications of Nyquist stability criterion.
- Gain and phase margins.
- Approximate relationship between phase margin and equivalent damping ratio.

Stability and Characteristic Equation

A system is Stable	$\Longleftrightarrow$	No pole has positive real part

Poles: The solutions (roots) of the characteristic equation.

Closed-loop transfer function (TF):

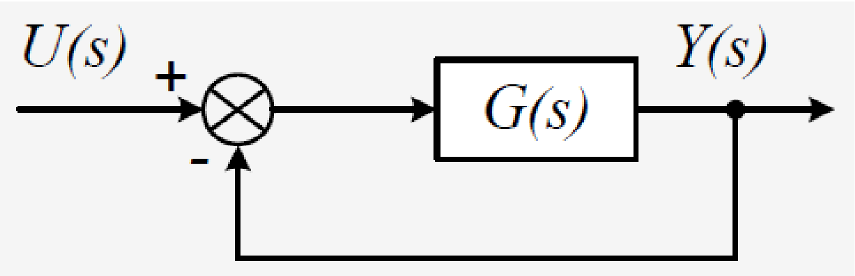


$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)} = \frac{N(s)}{D(s)+N(s)} \quad where \quad G(s) = \frac{N(s)}{D(s)}$$

The characteristic equation (CE)

$$D(s)+N(s)=0 \quad Equvalent\ to \quad 1+G(s)=0$$

Stable and Unstable System Examples



Open-loop TF: Stable

$$G(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

Poles:

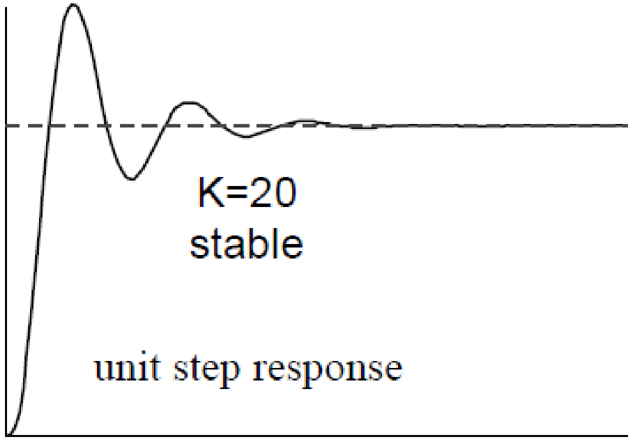
-1, -2, -3

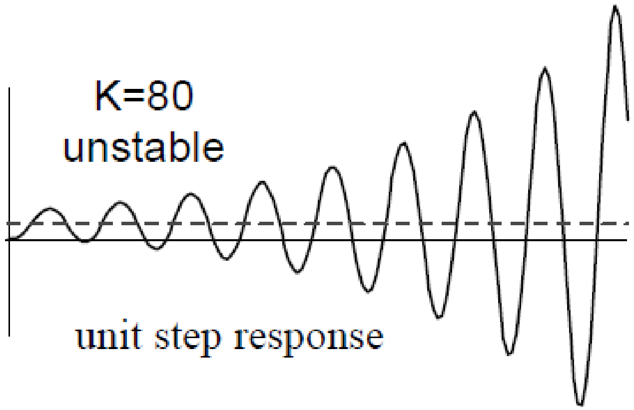
Closed-loop TF:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K}{(s+1)(s+2)(s+3)+K}$$

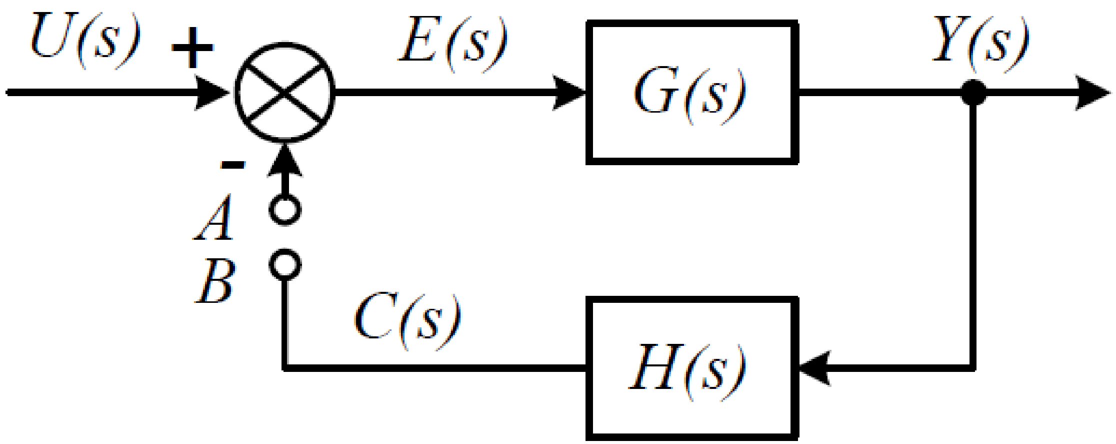
K=20, poles: *-4.84, -0.58±2.24j, stable*

K=80, poles: *-6.39, 0.19±3.66j, unstable*





General Characteristic Equation



- Closed-loop (link A-B) transfer function (CLTF):

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+GH(s)}$$

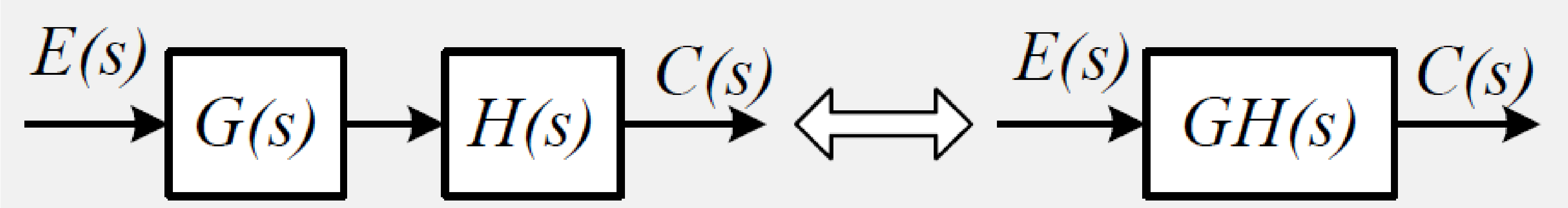
- The characteristic equation:

$$1+GH(s)=0 \quad or \quad GH(s)=-1$$

Open-loop TF and Closed-loop CE

$$\text{Closed-loop } CE: \quad 1 + GH(s) = 0$$

- The closed-loop (CL) system stability can be determined by $GH(s)$, which is defined as the open-loop (OL) transfer function (TF):



- In control systems analysis and design, the OLTF is normally known before the CLTF.
  - OLTF can be used to predict CL stability and performance.

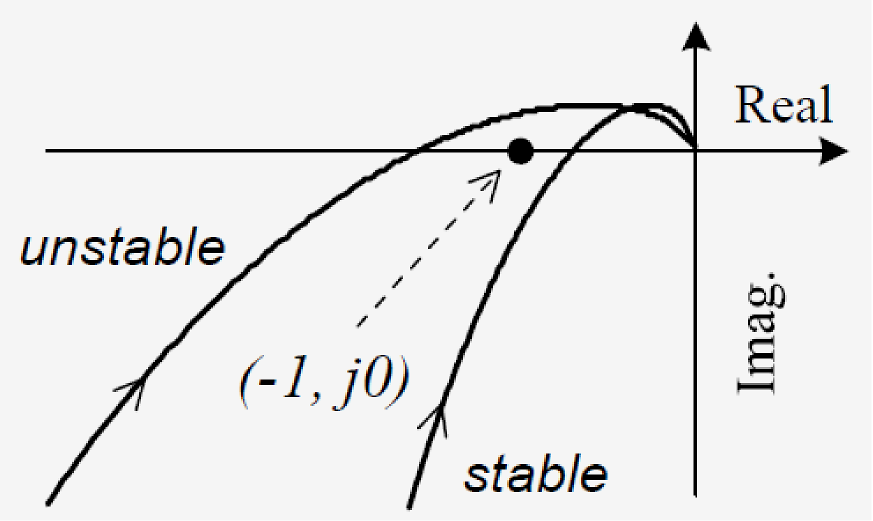
Check the CL stability using OL frequency response *GH(jω)*

CLCE: $1 + GH(s) = 0$

Frequency domain CLCE:

$$GH(j\omega) = -1 = -1 + j0 = 1\angle(-180°)$$

Check the relationship of OL polar plot with the point (-1, j0) on the complex plane



- ***<u>Nyquist stability criterion</u>***:
  - If the open-loop system is stable (no open-loop poles with positive real parts), the closed-loop system is stable *if and only if* the open-loop polar plot *GH(jω)* ***<u>does not encircle</u>*** the *(-1, j0)* point .

Example 1: Systems with 1st order OL TF

$$GH(s) = \frac{K}{Ts+1} \xrightarrow{s=j\omega} GH(j\omega) = \frac{K}{jT\omega+1}$$

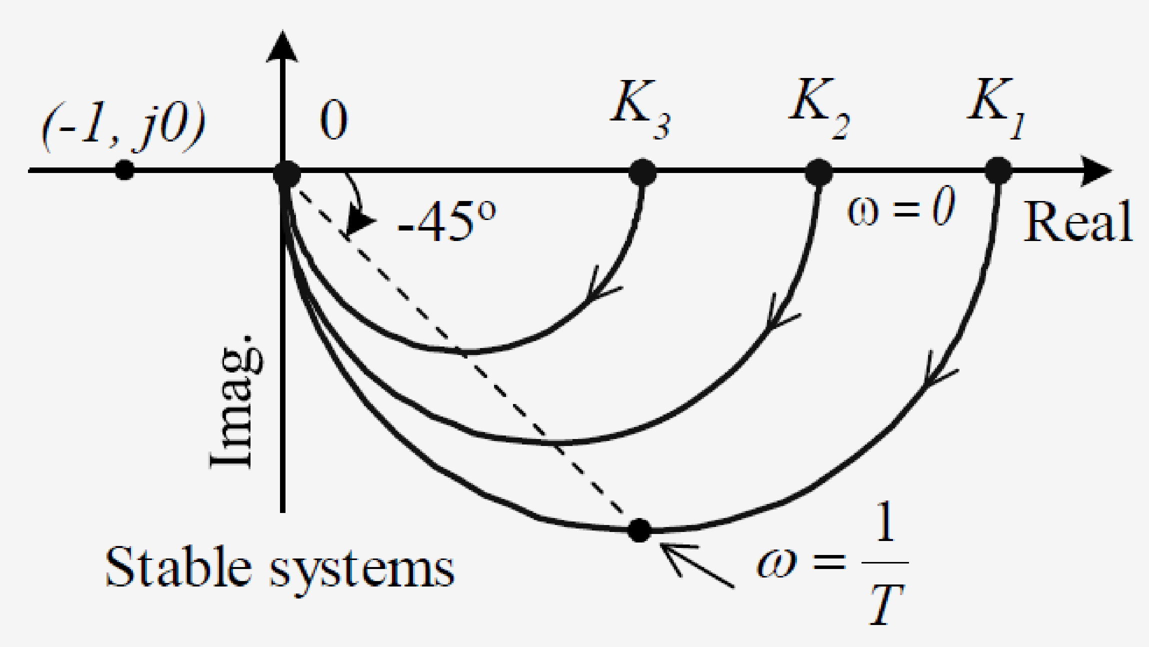
$$M(\omega) = |GH(j\omega)| = \frac{K}{\sqrt{1+(T\omega)^2}}$$

$$\phi(\omega) = \angle GH(j\omega) = -\tan^{-1} T\omega$$

$$\omega = 0 \quad M(\omega) = K \quad \phi(\omega) = 0°$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -90°$$

- The polar plot does not encircle the point *(-1, j0)* when ω varies from 0 to +∞, the CL system is always stable.



Example 2: Systems with 2nd order OL TF

$$GH(s) = \frac{K}{s(Ts+1)} \xrightarrow{s=j\omega} GH(j\omega) = \frac{K}{j\omega(jT\omega+1)}$$

$$M(\omega) = |GH(j\omega)| = \frac{K}{\omega\sqrt{1+(T\omega)^2}}$$

$$\phi(\omega) = \angle GH(j\omega) = -90^o - \tan^{-1} T\omega$$

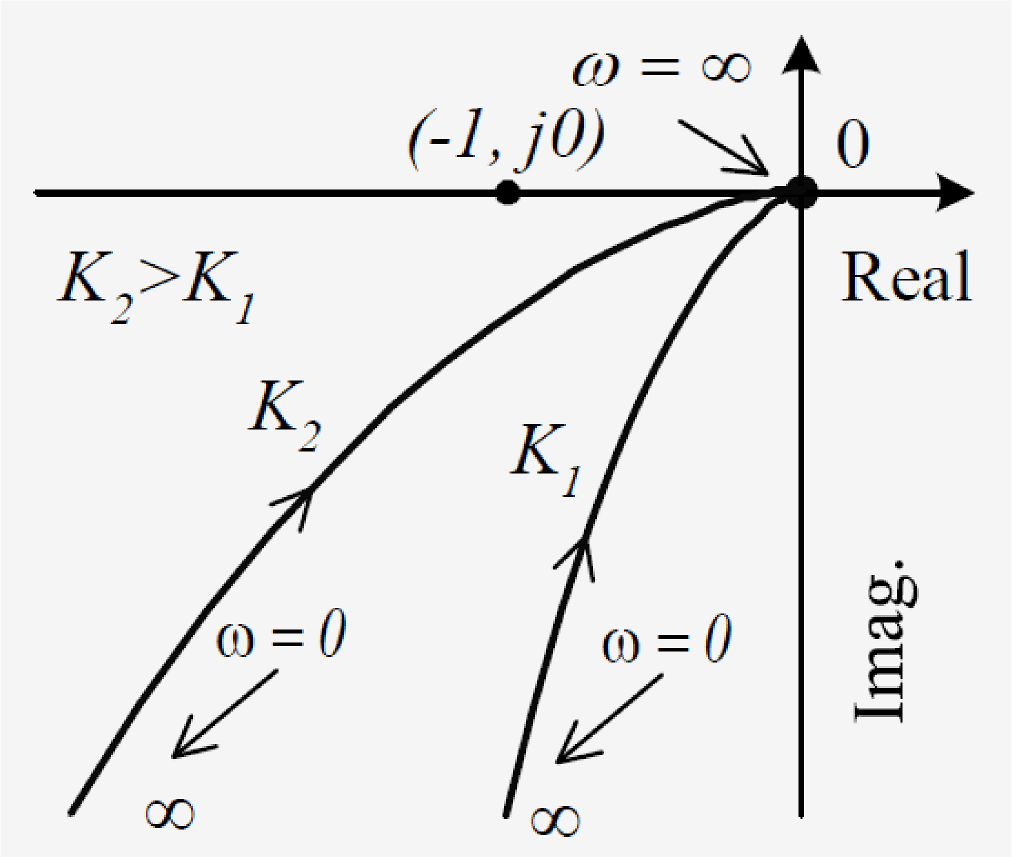
$$\omega = 0 \quad M(\omega) = \infty \quad \phi(\omega) = -90^\circ$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -180^\circ$$

Example 2: Systems with 2nd order OL TF

$$\omega = 0 \quad M(\omega) = \infty \quad \phi(\omega) = -90^\circ$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -180^\circ$$



- The polar plot does not encircle the point *(-1, j0)* when ω varies from 0 to +∞, the CL system is always stable.

Stable Systems

Example 3: Systems with 3rd order OL TF

$$GH(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}$$

$s = j\omega$

$$GH(j\omega) = \frac{K}{j\omega(jT_1\omega + 1)(jT_2\omega + 1)}$$

$$M(\omega) = |GH(j\omega)| = \frac{K}{\omega\sqrt{1 + (T_1\omega)^2}\sqrt{1 + (T_2\omega)^2}}$$

$$\phi(\omega) = \angle GH(j\omega) = -90^o - \tan^{-1} T_1\omega - \tan^{-1} T_2\omega$$

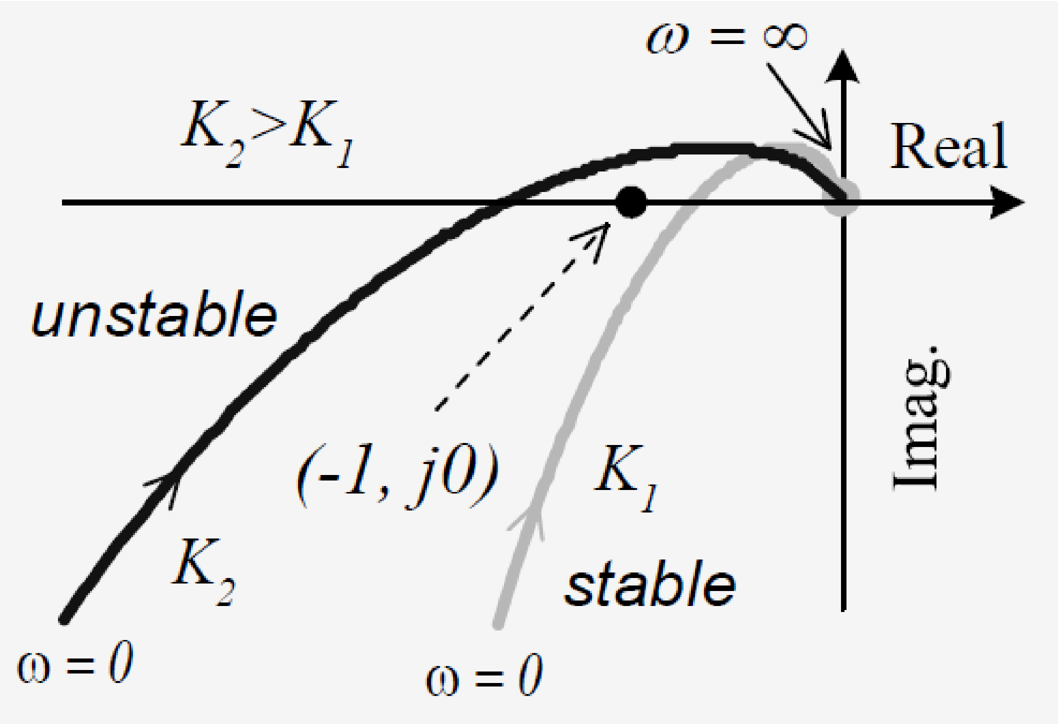
$$\omega = 0 \quad M(\omega) = \infty \quad \phi(\omega) = -90^\circ$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -270^\circ$$

Example 3: Systems with 3rd order OL TF

$$\omega = 0 \quad M(\omega) = \infty \quad \phi(\omega) = -90^\circ$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -270^\circ$$



- The CL system may be unstable when the *K* becomes bigger!
  - find *K* when polar plot just passes through *(-1, j0)*

Conditionally Stable Systems

Example 4: Systems with 3rd order OL TF

$$GH(s) = \frac{K}{s^2(Ts+1)}$$

$$s = j\omega$$

$$GH(j\omega) = \frac{K}{(j\omega)^2(jT\omega+1)}$$

$$M(\omega) = \left|GH(j\omega)\right| = \frac{K}{\omega^2\sqrt{1+(T\omega)^2}}$$

$$\phi(\omega) = \angle GH(j\omega) = -180^o - \tan^{-1} T\omega$$

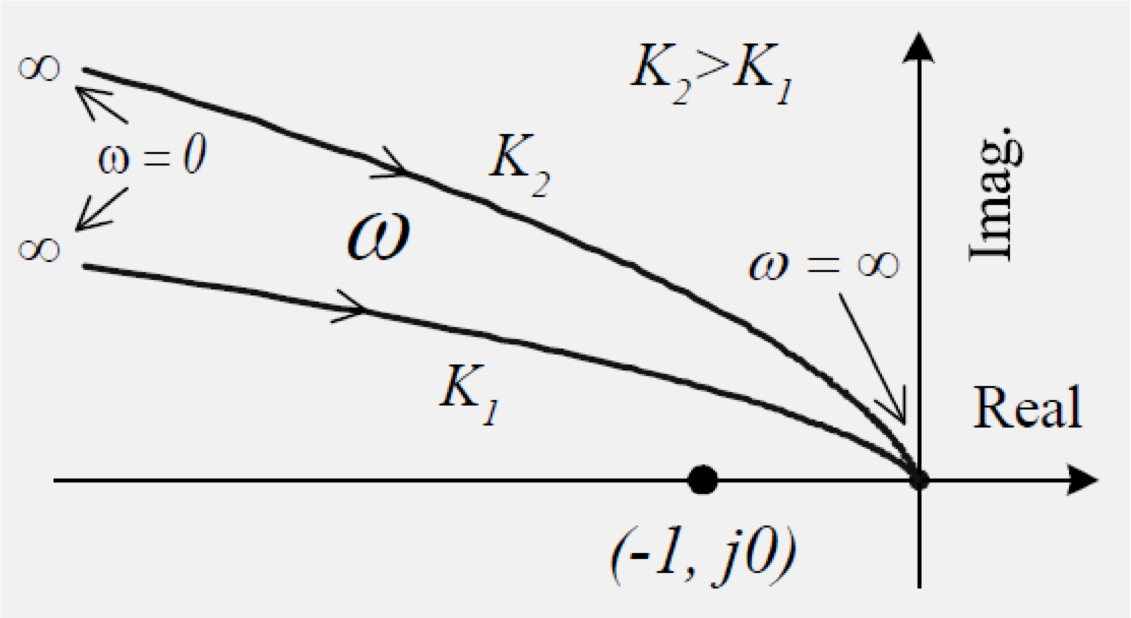
$$\omega = 0 \quad M(\omega) = \infty \quad \phi(\omega) = -180^\circ$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -270^\circ$$

Example 4: Systems with 3rd order OL TF

$$\omega = 0 \quad M(\omega) = \infty \quad \phi(\omega) = -90^\circ$$

$$\omega = \infty \quad M(\omega) = 0 \quad \phi(\omega) = -270^\circ$$



- The polar plot encircles the point *(-1, j0)* when ω varies from 0 to +∞, the CL system is always unstable.

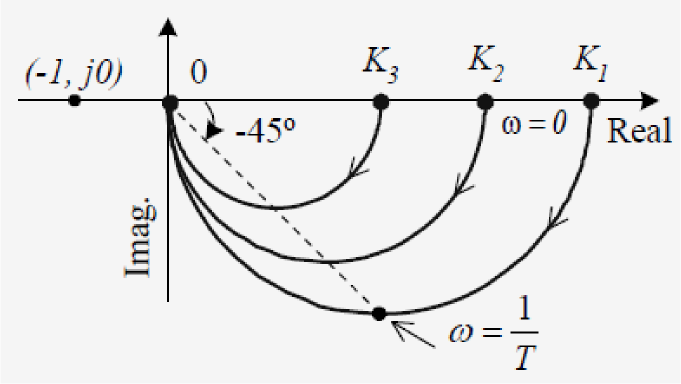
Unstable Systems

Stability Cases

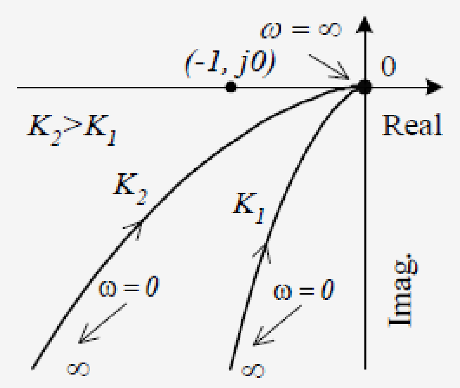
Systems with stable OLTF, CL system maybe stable, maybe unstable.

Example 1: *$0^o \rightarrow -90^o$*

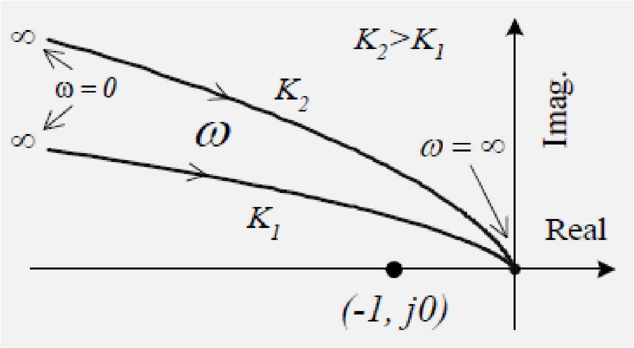
Stable when phase *larger* than -180°



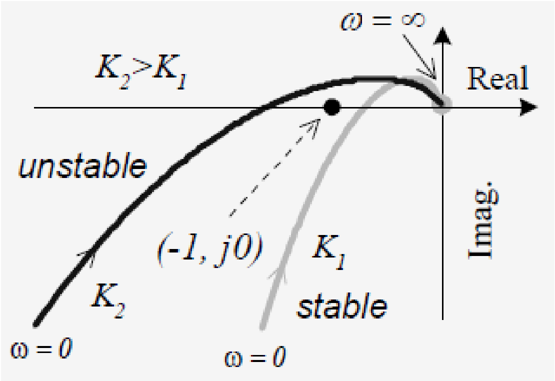
Example 2: *$-90^o \rightarrow -180^o$*



Unstable when phase *smaller* than -180°



Example 4: *$-180^o \rightarrow -270^o$*



Stable if gain less than 1 when the phase is *-180^o*

Stable/ unstable if the phase passes through *-180^o*

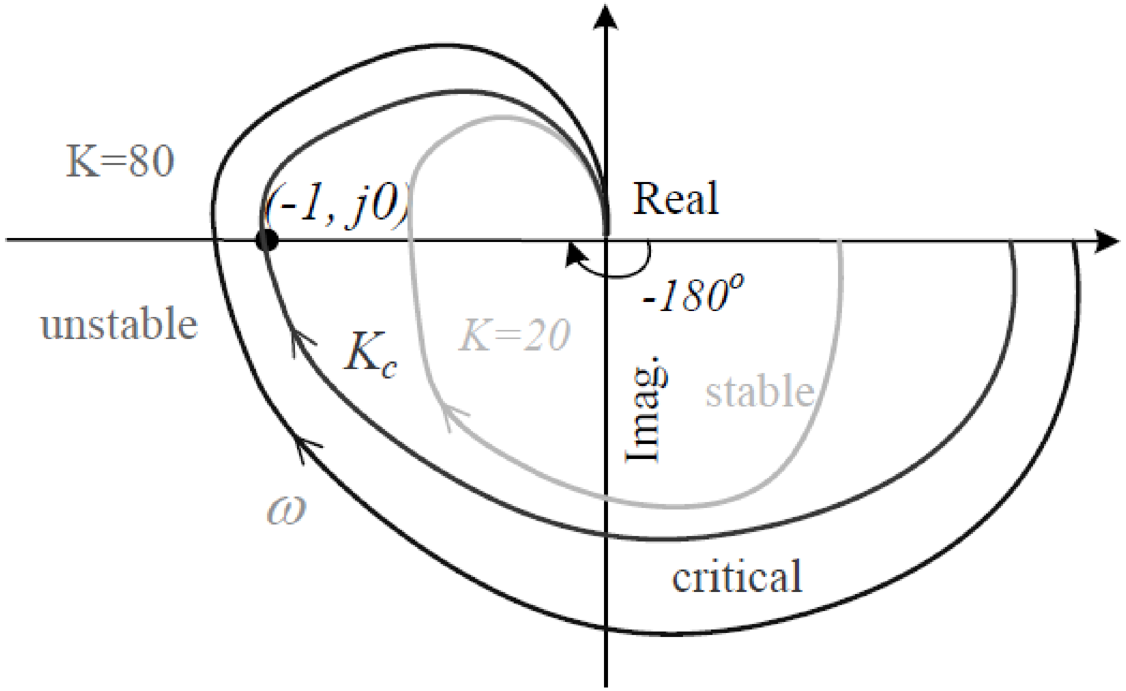
Nyquist Stability Example

Find the critical value for K when the CL system becomes unstable for the system with the following OLTF:

$$GH(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

Open loop is stable: no poles with positive real parts.

Closed Loop is conditionally stable.



Find the critical K for the limit of stability

- Find the value of K when the polar plot passes through the point (-1, $j0$) on the complex plane.

When the polar plot passes through the point (-1, $j0$) on the complex plane

$$GH(j\omega) = -1 = -1 + j0$$
$$= 1\angle(-180°)$$

Step 1: set $\angle GH(j\omega) = -180^o$ and solve for $\omega = \omega_c$

Step 2: substitute $\omega = \omega_c$ in $|GH(j\omega)| = 1$ to find K_c

Nyquist Stability Example

$$GH(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

$$GH(j\omega) = \frac{K}{(j\omega+1)(j\omega+2)(j\omega+3)}$$

$$\begin{aligned}
&\angle GH(j\omega) \\
&= \angle\left(K\right) - \angle\left((j\omega+1)(j\omega+2)(j\omega+3)\right) \\
&= 0^{o} - \left(\angle(j\omega+1) + \angle(j\omega+2) + \angle(j\omega+3)\right) \\
&= -\tan^{-1}\omega - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{3}
\end{aligned}$$

Nyquist Stability Example

Step 1: Firstly, find the ω_c for which the phase is -180^o.

$$\angle GH(j\omega_c) = -\tan^{-1}\omega_c - \tan^{-1}\frac{\omega_c}{2} - \tan^{-1}\frac{\omega_c}{3} = -180^o$$

$$\tan^{-1}\omega_c + \tan^{-1}\frac{\omega_c}{2} = 180^o - \tan^{-1}\frac{\omega_c}{3}$$

$$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$$

$$\tan^{-1}\frac{\omega_c + 0.5\omega_c}{1-\omega_c \times 0.5\omega_c} = 180^o - \tan^{-1}\frac{\omega_c}{3}$$

$$\frac{1.5\omega_c}{1-0.5\omega_c^2} = \tan\left(180^o - \tan^{-1}\frac{\omega_c}{3}\right)$$

Nyquist Stability Example

$$\frac{1.5\cancel{\omega}_c}{1-0.5\omega_c^2} = -\tan\left(\tan^{-1}\frac{\omega_c}{3}\right) = -\frac{\cancel{\omega}_c}{3}$$

$$\frac{1.5}{1-0.5\omega_c^2} = -\frac{1}{3} \Rightarrow 1-0.5\omega_c^2 = -4.5$$

$$0.5\omega_c^2 = 5.5 \Rightarrow \omega_c = \sqrt{11} = 3.32$$

Nyquist Stability Example

Step 2: Substitute ω_c in $|GH(j\omega_c)| = 1$ to find K. $\omega_c = \sqrt{11}$

$$|GH(j\omega)| = \left| \frac{K}{(j\omega+1)(j\omega+2)(j\omega+3)} \right|$$

$$= \frac{|K|}{|(j\omega+1)(j\omega+2)(j\omega+3)|} = \frac{|K|}{|j\omega+1||j\omega+2||j\omega+3|}$$

$$|GH(j\omega_c)| = \frac{K}{\sqrt{1^2+\omega_c^2}\sqrt{2^2+\omega_c^2}\sqrt{3^2+\omega_c^2}}$$

$$= \frac{K}{\sqrt{1+11}\sqrt{4+11}\sqrt{9+11}}$$

$$= \frac{K}{\sqrt{12}\sqrt{15}\sqrt{20}} = \frac{K}{\sqrt{3600}} = 1 \quad \Rightarrow \quad K = 60$$

Stable when $K<60$

Nyquist stability criterion using Bode plot

- The CL system is stable if the gain is less than *0dB* when the phase is -180^o .