Instrumentation and Controls
ETM 3301

Lecture 22

Instructor

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Second Order System Resonant Frequency and Resonant Peak

Gain: $$M(\omega) = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}$$

$M(\omega)$ maximal equivalent to $P(\omega)$ minimal.

$$P(\omega) = (\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2$$

$$\frac{dP(\omega)}{d\omega} = 4\omega\left[\omega^2 - (1-2\zeta^2)\omega_n^2\right] = 0 \Rightarrow \omega^2 = (1-2\zeta^2)\omega_n^2$$

When $1-2\zeta^2 > 0$ or $\zeta < \frac{1}{\sqrt{2}} = 0.707$ A solution exists.

Second Order System Resonant Frequency and Resonant Peak, 2

$$\omega^2 = \left(1 - 2\zeta^2\right)\omega_n^2$$

When $\zeta < 0.707$, the 2nd order system has resonance and the resonant frequency and resonant peak are:

Resonant frequency: $\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$

Resonant peak: $M_p = M(\omega_r) = \dfrac{1}{2\zeta\sqrt{1 - \zeta^2}}$

Resonant Frequency and Resonant Peak Example

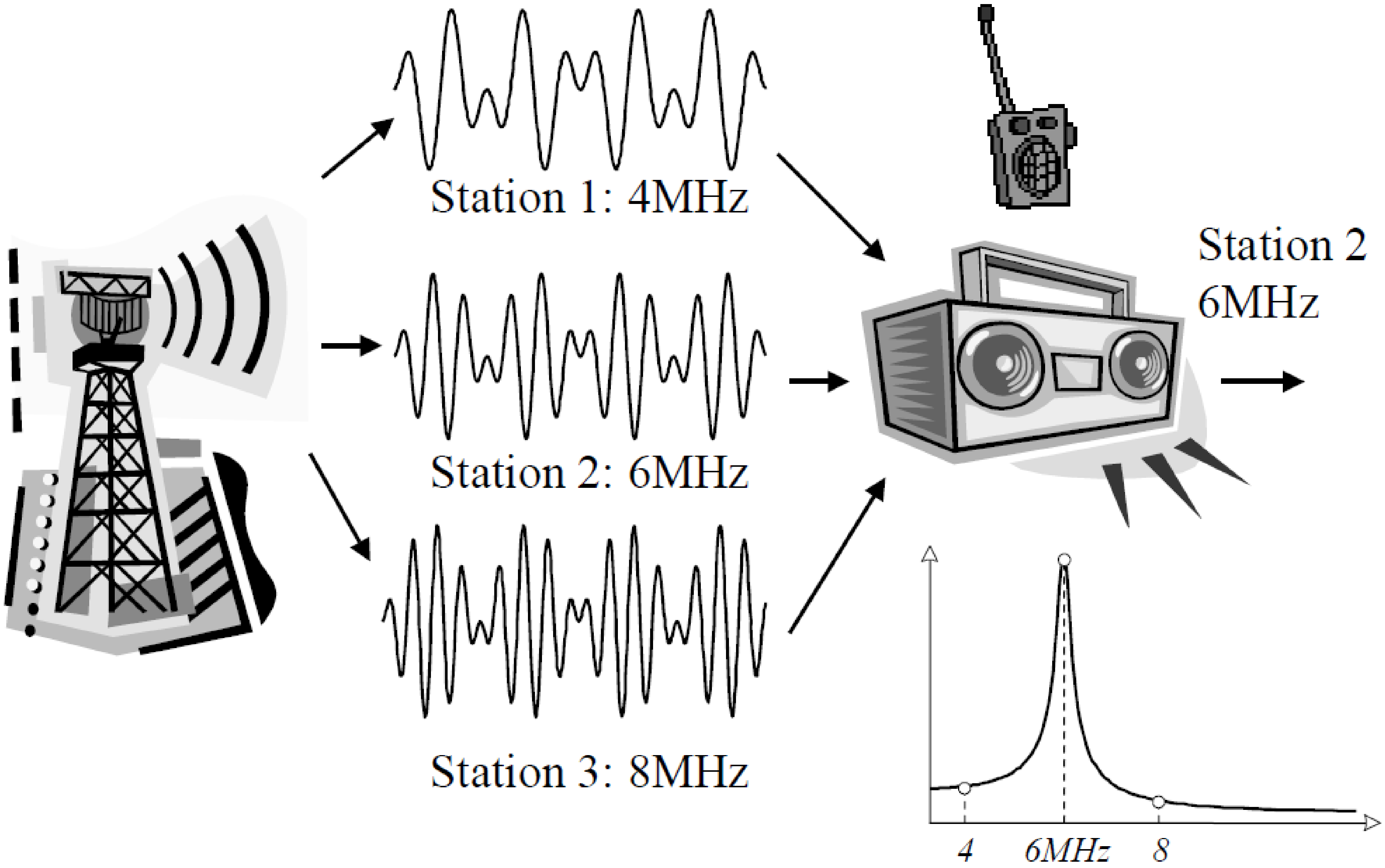
$$G(s) = \frac{25}{s^2 + 2s + 25} \qquad \left.\begin{array}{r} \omega_n^2 = 25 \\ 2\zeta\omega_n = 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \omega_n = 5 \\ \zeta = 0.2 \end{array}\right.$$

$\zeta = 0.2 < 0.707$, resonance exists

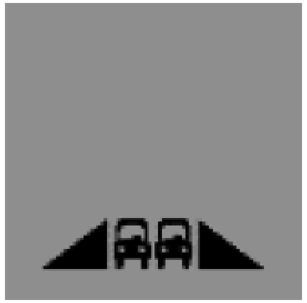
Resonant frequency: $\omega_r = \omega_n\sqrt{1-2\zeta^2} = 4.899\ rad/s$

Resonant peak: $M_p = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = 2.552$

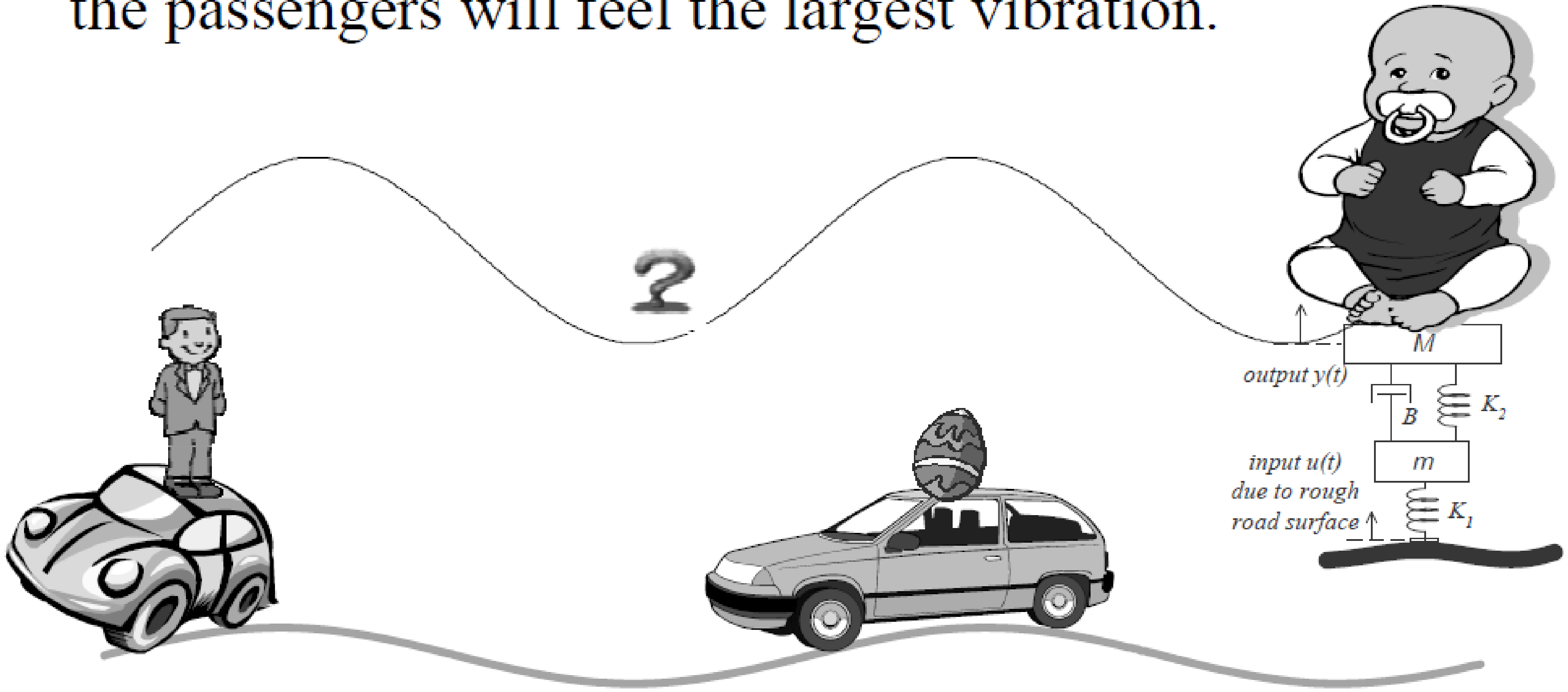
Resonance is "*Good*"



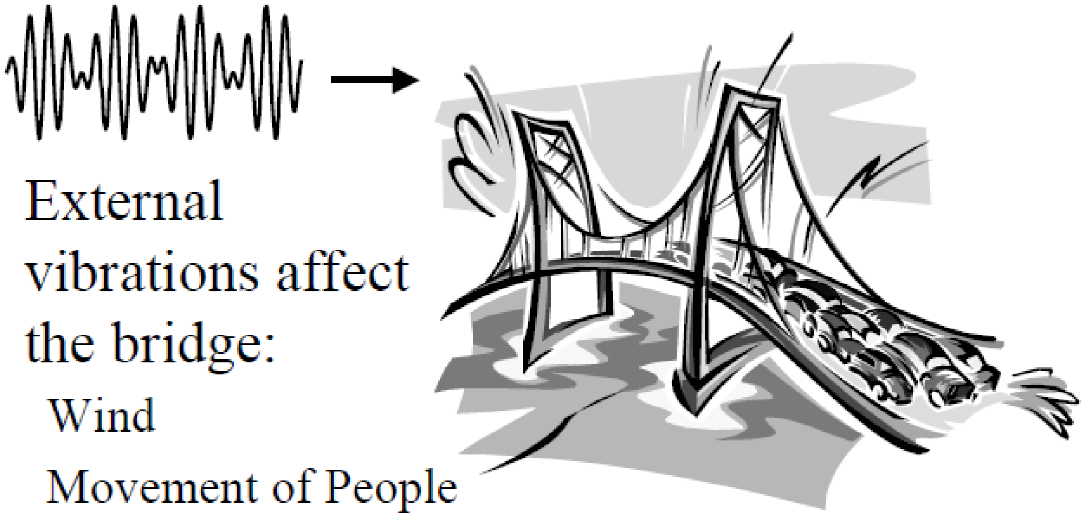
Resonance is "Bad"



If the resonant frequency of the car's suspension system is the same as the vibration frequency caused by bumpy road, the passengers will feel the largest vibration.



Resonance is "Ugly (Disaster)"

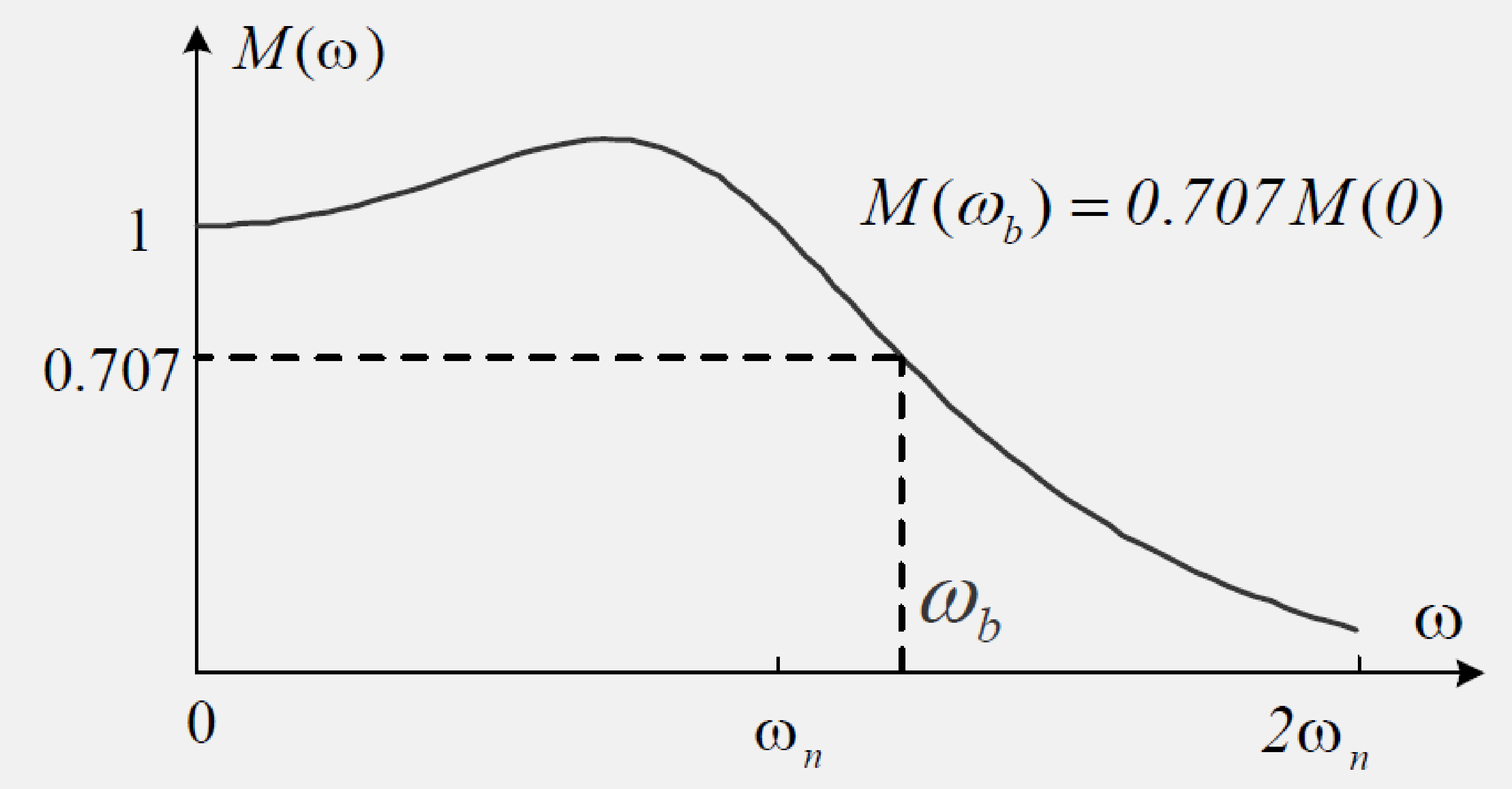


External vibrations affect the bridge:

- Wind
- Movement of People and vehicles

BBC: Swaying Millennium Bridge to close after two days: The suspension bridge swayed alarmingly under the weight of thousands of people at its official opening on Saturday.

Bandwidth



- **Bandwidth** ω_b: The bandwidth ω_b is the frequency at which the gain $M(\omega)$ drops to 0.707 of its zero-frequency value.

2nd order system: $\omega_b = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{\zeta^4-4\zeta^2+2}}$

More Frequency Response Example

$$G(s) = \frac{(3s+1)}{s(s+1)(2s+1)} \qquad G(j\omega) = \frac{(3j\omega+1)}{j\omega(j\omega+1)(2j\omega+1)}$$

Gain:

$$|G(j\omega)| = \left|\frac{(3j\omega+1)}{j\omega(j\omega+1)(2j\omega+1)}\right| = \frac{|(3j\omega+1)|}{|j\omega(j\omega+1)(2j\omega+1)|}$$

$$= \frac{|j3\omega+1|}{|j\omega||j\omega+1||j2\omega+1|} = \frac{\sqrt{(3\omega)^2+1^2}}{\omega\sqrt{\omega^2+1^2}\sqrt{(2\omega)^2+1^2}}$$

$$= \frac{\sqrt{9\omega^2+1}}{\omega\sqrt{\omega^2+1}\sqrt{4\omega^2+1}}$$

More Frequency Response Example

$$G(s) = \frac{(3s+1)}{s(s+1)(2s+1)} \qquad G(j\omega) = \frac{(3j\omega+1)}{j\omega(j\omega+1)(2j\omega+1)}$$

Phase:

$$\angle(j\omega) = 90^o$$

$$\angle G(j\omega) = \angle\left(\frac{(3j\omega+1)}{j\omega(j\omega+1)(2j\omega+1)}\right)$$

$$= \angle(3j\omega+1) - \angle\left(j\omega(j\omega+1)(2j\omega+1)\right)$$

$$= \angle(3j\omega+1) - \left[\angle(j\omega) + \angle(j\omega+1) + \angle(2j\omega+1)\right]$$

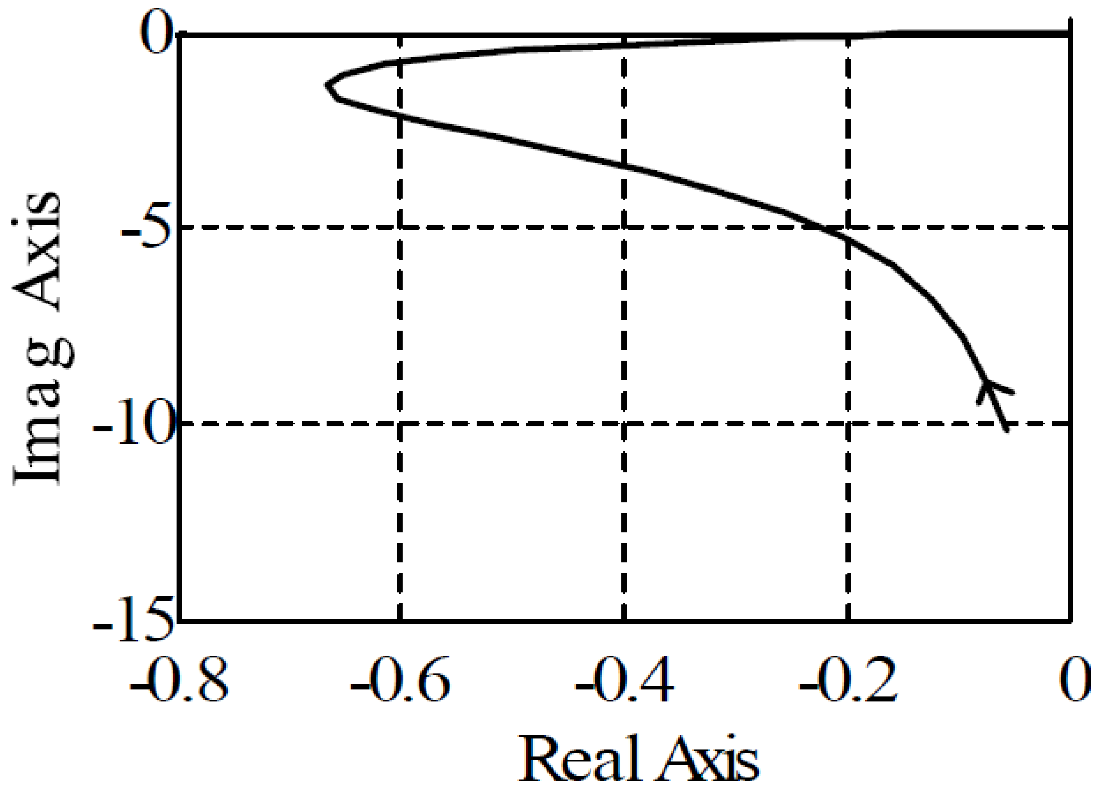
$$= \tan^{-1}\frac{3\omega}{1} - 90^o - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{2\omega}{1}$$

$$= \tan^{-1} 3\omega - 90^o - \tan^{-1}\omega - \tan^{-1} 2\omega$$

More Frequency Response Example

$$|G(j\omega)| = \frac{\sqrt{9\omega^2 + 1}}{\omega\sqrt{\omega^2 + 1}\sqrt{4\omega^2 + 1}}$$

$$\angle G(j\omega) = \tan^{-1} 3\omega - 90^o - \tan^{-1}\omega - \tan^{-1} 2\omega$$



Frequency response for series elements

$$G(s) = G_1(s)G_2(s)G_3(s)$$

Frequency response

$$G(j\omega) = G_1(j\omega)G_2(j\omega)G_3(j\omega)$$

Gain

$$|G(j\omega)| = |G_1(j\omega)||G_2(j\omega)||G_3(j\omega)|$$

Phase:

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \angle G_3(j\omega)$$

Frequency response for series elements: Example

$$G(s) = \frac{1}{(s+1)(2s+1)} = \frac{1}{s+1} \times \frac{1}{2s+1} = G_1(s)G_2(s)$$

$$G_1(j\omega) = \frac{1}{j\omega + 1} \qquad G_2(j\omega) = \frac{1}{j2\omega + 1}$$

$$M_1(\omega) = \frac{1}{\sqrt{\omega^2 + 1}} \qquad M_2(\omega) = \frac{1}{\sqrt{4\omega^2 + 1}}$$

$$\phi_1(\omega) = -\tan^{-1}\omega \qquad \phi_2(\omega) = -\tan^{-1} 2\omega$$

Gain: $$M(\omega) = M_1 M_2 = \frac{1}{\sqrt{\omega^2 + 1}} \times \frac{1}{\sqrt{4\omega^2 + 1}}$$

Phase: $$\phi(\omega) = \phi_1(\omega) + \phi_2(\omega) = -\tan^{-1}\omega - \tan^{-1} 2\omega$$

Gain response for series elements

$$G(s) = G_1(s)G_2(s)G_3(s)$$

$$G(j\omega) = G_1(j\omega)G_2(j\omega)G_3(j\omega)$$

Gain $$|G(j\omega)| = |G_1(j\omega)||G_2(j\omega)||G_3(j\omega)|$$

Phase $$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \angle G_3(j\omega)$$

Logarithmic gain

$$\begin{aligned} 20\log_{10}|G(j\omega)| &= 20\log_{10}|G_1(j\omega)| \\ &\quad + 20\log_{10}|G_2(j\omega)| + 20\log_{10}|G_3(j\omega)| \end{aligned}$$

Gain response for series elements

$$G(s)=\frac{1}{(s+1)(2s+1)}=\frac{1}{s+1}\times\frac{1}{2s+1}$$

Gain

$$20\log_{10}\frac{1}{\sqrt{\omega^2+1}}+20\log_{10}\frac{1}{\sqrt{4\omega^2+1}}$$

$$=-10\log_{10}(\omega^2+1)-10\log_{10}(4\omega^2+1)$$