# Differential Equations

The first order differential equations.

$$\frac{dv}{dt} = f(v)\, g(t)$$

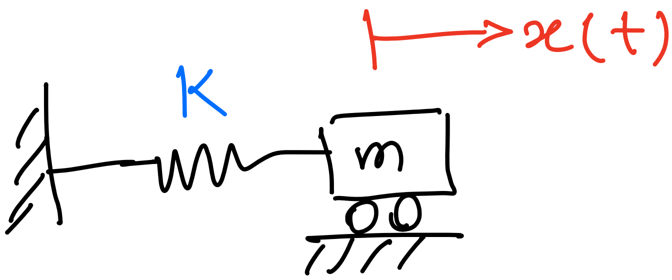
For the first order differential equations seperate the variables on each side of the equation, and integrate

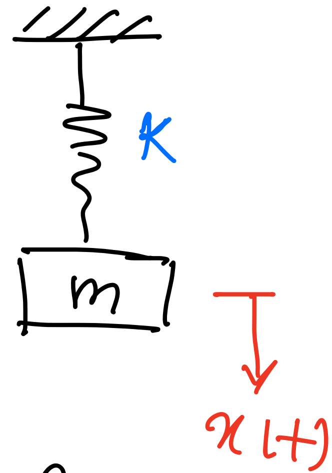
$$\frac{dv}{f(v)} = g(t)\, dt$$

$$\Rightarrow \int \frac{1}{f(v)}\, dv = \int g(t)\, dt$$

# 2nd order differential equations

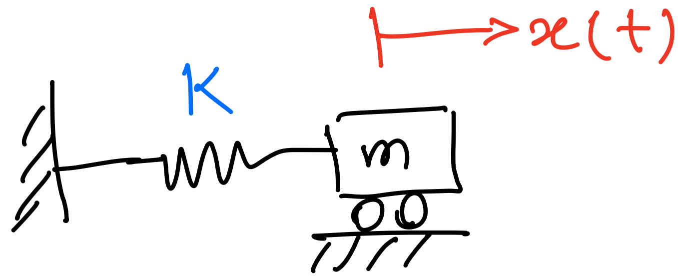
Free vibration of a mass-spring system is shown in the example below.



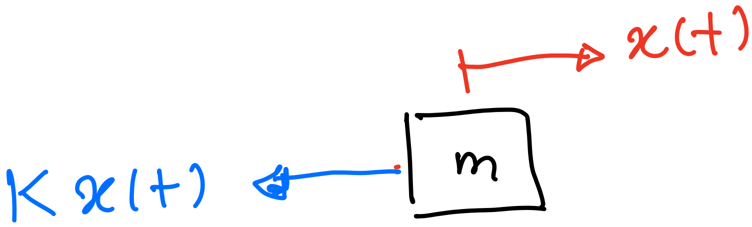




If the initial displacement of $x(0)$ at time equal to zero is $x(0) = A$, and the initial velocity is zero $(\dot{x}(0) = 0)$ (Initial means time $t = 0$).

Find the response of the mass $x(t) = ?$




freebody diagram (F.B.D.)




$$\overset{+}{\rightarrow} \sum F = m\ddot{x} \implies -Kx(t) = m\ddot{x}(t)$$

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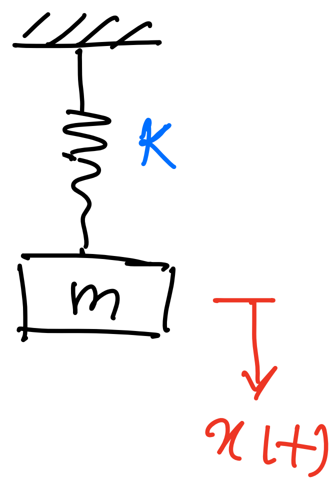
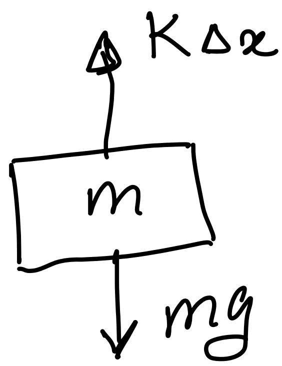
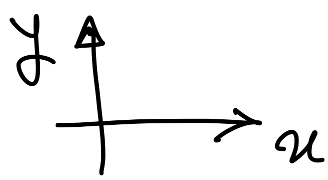
Static:
Mass is not moving




FBD
for static
equilibrium



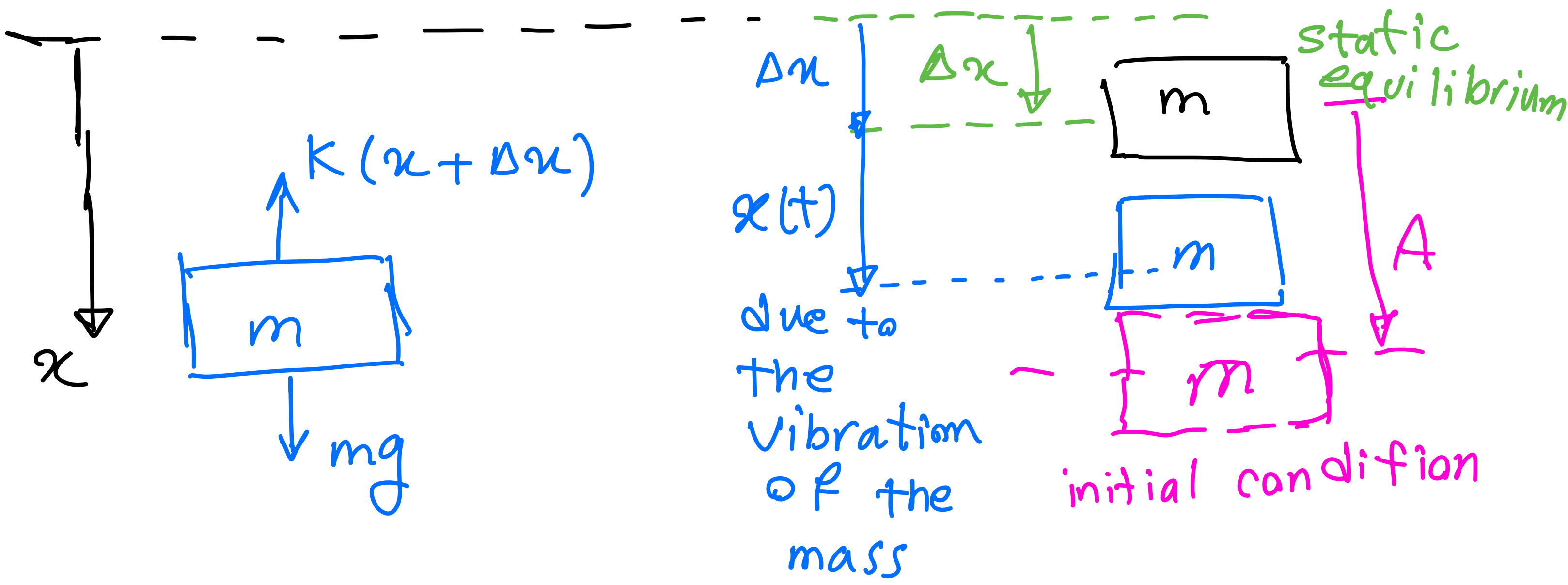




$$+\uparrow \sum F_y = 0$$

$$K\Delta x - mg = 0 \implies K\Delta x = mg \quad *$$

Repeat the problem for dynamics
when mass m is moving.

F.B.D (Dynamics: when the mass is moving)




$$+\uparrow \sum F = m\ddot{x}(t)$$

$$K(x+\Delta x) - mg = -m\ddot{x}(t)$$

$$Kx + K\underbrace{\Delta x - mg}_{=0} = -m\ddot{x}(t)$$

$*$

$$Kx = -m\ddot{x}(t)$$

$$m\ddot{x}(t) + Kx = 0$$

The Equation of motion is:

$$-Kx(t) = m\ddot{x}(t)$$

$$\boxed{\ddot{x} = \frac{d^2x}{dt^2} \quad \text{Acceleration}}$$

Rearrang:

$$m\ddot{x}(t) + Kx(t) = 0$$

Let's assume that the solution is in the following form:

$$x(t) = Ce^{st}$$

"s" is a constant

"C" is a constant

If we find "C" and "S" then we know the solution.

we can substitute this solution into the equation of motion:

$$m\ddot{x}(t) + Kx(t) = 0$$

$$x(t) = Ce^{st} \Longrightarrow \ddot{x}(t) = Cs^2 e^{st}$$

substitute:

$$m\cancel{C}s^2 \cancel{e^{st}} + K\cancel{C}\cancel{e^{st}} = 0$$

$$ms^2 + K = 0 \Longrightarrow s^2 = -K/m$$

$$\Longrightarrow s = \pm\sqrt{-K/m}$$

$$\Longrightarrow s = \pm i\sqrt{\frac{K}{m}}$$

$$\begin{cases} s_1 = i\sqrt{\frac{K}{m}} \\ s_2 = -i\sqrt{K/m} \end{cases} \quad (i = \sqrt{-1})$$

$$x(t) = Ce^{st}$$

$$x(t) = Ce^{\pm i\sqrt{\frac{K}{m}}\,t}$$

$$x(t) = c_1 e^{i\sqrt{\frac{k}{m}}t} + c_2 e^{-i\sqrt{k/m}\,t}$$

Initial conditions :

$x(0) = A \qquad \dot{x}(0) = 0$

substitute into $x(t)$ .

$$x(0) = c_1 + c_2 = A$$

$$e^0 = 1$$

$$\dot{x}(0) = 0$$

$$\dot{x}(t) = c_1 i\sqrt{\frac{k}{m}}\, e^{i\sqrt{k/m}\,t} - c_2 i\sqrt{k/m}\, e^{-i\sqrt{\frac{k}{m}}t}$$

$$\dot{x}(0) = c_1 i \cancel{\sqrt{\frac{k}{m}}} - c_2 i \cancel{\sqrt{\frac{k}{m}}} = 0$$

$$\begin{cases} c_1 + c_2 = A \\ c_1 - c_2 = 0 \Rightarrow c_1 = c_2 \end{cases}$$

$$\boxed{c_1 = c_2 = A/2}$$

Therefore the solution is:

$$x(t) = c_1 e^{i\sqrt{k/m}\,t} + c_2 e^{-i\sqrt{k/m}\,t}$$

$$x(t) = \frac{A}{2} e^{i\sqrt{k/m}\,t} + \frac{A}{2} e^{-i\sqrt{\frac{k}{m}}\,t}$$

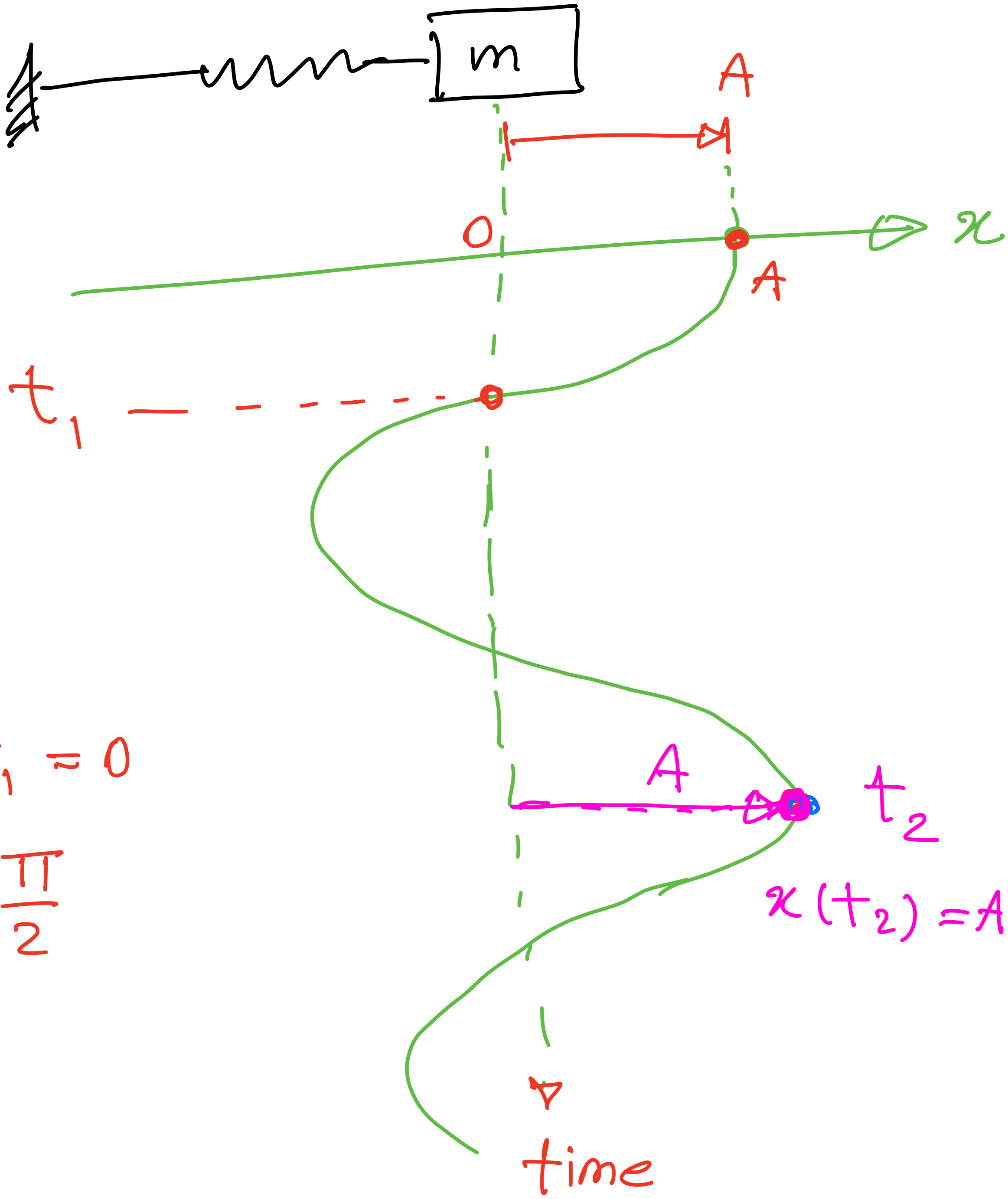
we can rewrite $x(t)$ as a function of sin and cos:

$e^{i\theta} = \cos\theta + i\sin\theta$

$$x(t) = \frac{A}{2}\left[\cos\sqrt{\frac{k}{m}}\,t \cancel{+ i\sin\sqrt{\frac{k}{m}}\,t}\right]$$

$$+ \left[\cos\sqrt{\frac{k}{m}}\,t \cancel{- i\sin\sqrt{\frac{k}{m}}\,t}\right]$$

$$x(t) = A\cos\sqrt{\frac{k}{m}}\,t$$




at $t_1$

$x(t)=0$

$A \cos\sqrt{\frac{k}{m}}\, t_1 = 0$

$\sqrt{\frac{k}{m}}\, t_1 = \frac{\pi}{2}$

$t_1 = \frac{\pi}{2\sqrt{\frac{k}{m}}}$

$x(t_2) = A$

$A \cos\sqrt{\frac{k}{m}}\, t_2 = A$

$\cos\sqrt{\frac{k}{m}}\, t_2 = 1$

$$\Rightarrow \sqrt{\frac{K}{m}}\, t_2 = 2\pi$$

$$t_2 = \frac{2\pi}{\sqrt{K/m}}$$

The time for one full cycle:

$t_2 = T$ (The period of oscillation

$f$ (frequency in Hz) $= \frac{1}{T}$

$$f = \frac{1}{\frac{2\pi}{\sqrt{K/m}}} = \frac{\sqrt{K/m}}{2\pi}$$

Angular frequency:

$\omega = 2\pi f$ (rad/s)

$$\Longrightarrow \omega = \sqrt{\frac{K}{m}} \quad \text{rad/s}$$