Instrumentation and Controls

ETM 3301

Lecture 20

Instructor

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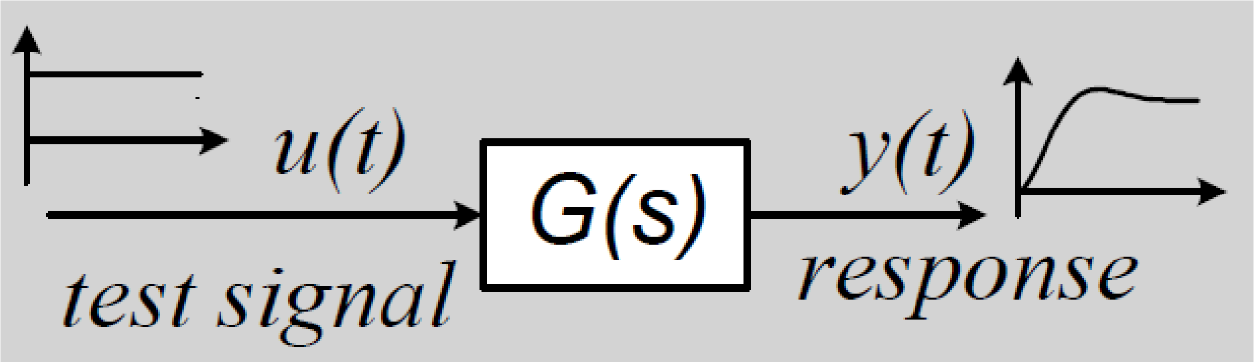
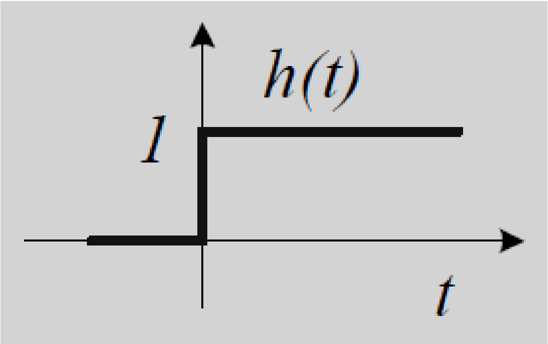
Chapter 8: Introduction to Frequency Responses

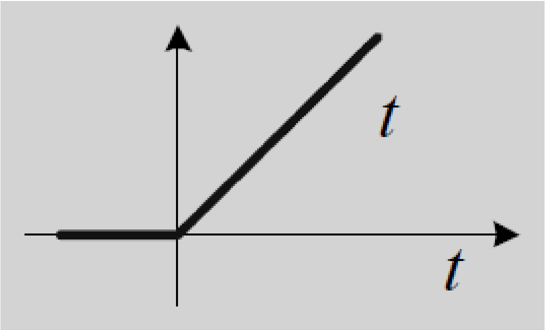
- Linear system responses to sinusoidal input signals.
- What is a frequency response and how to determine it?
  - Experiment method
  - Theoretical method
- Frequency responses of 1st and 2nd order systems.
- Resonance and resonate frequency.

System Test Using Test Signals

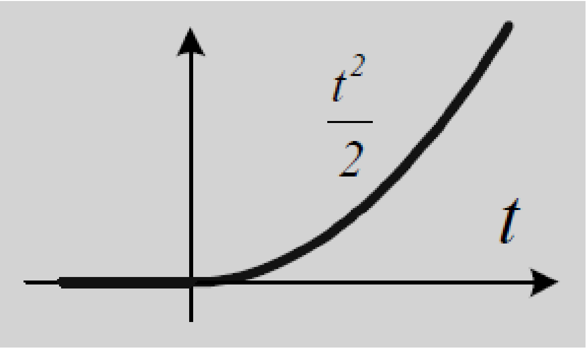




Step Signal



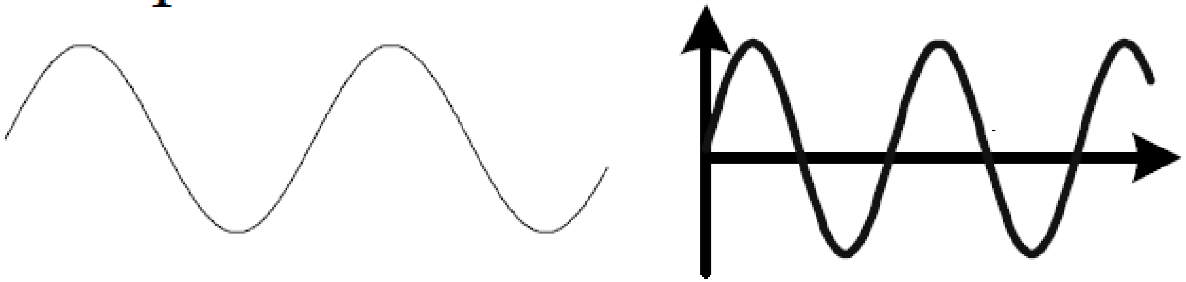
Ramp Signal



Parabolic Signal

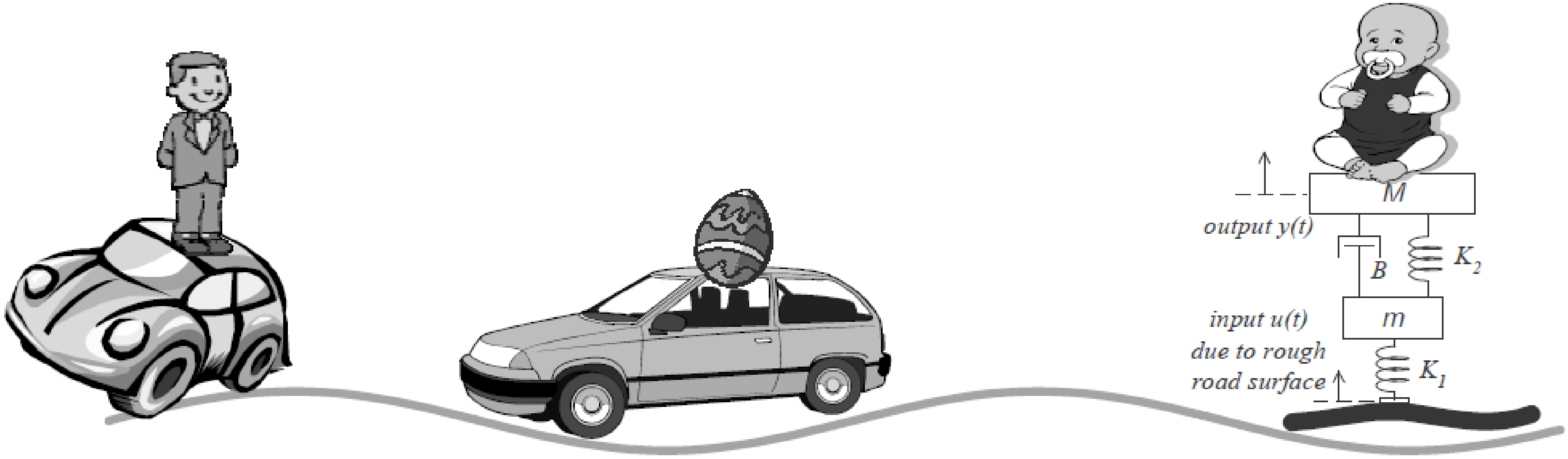
System Test Using Test Signals

To test the car’s suspension system performance due to the road profile.

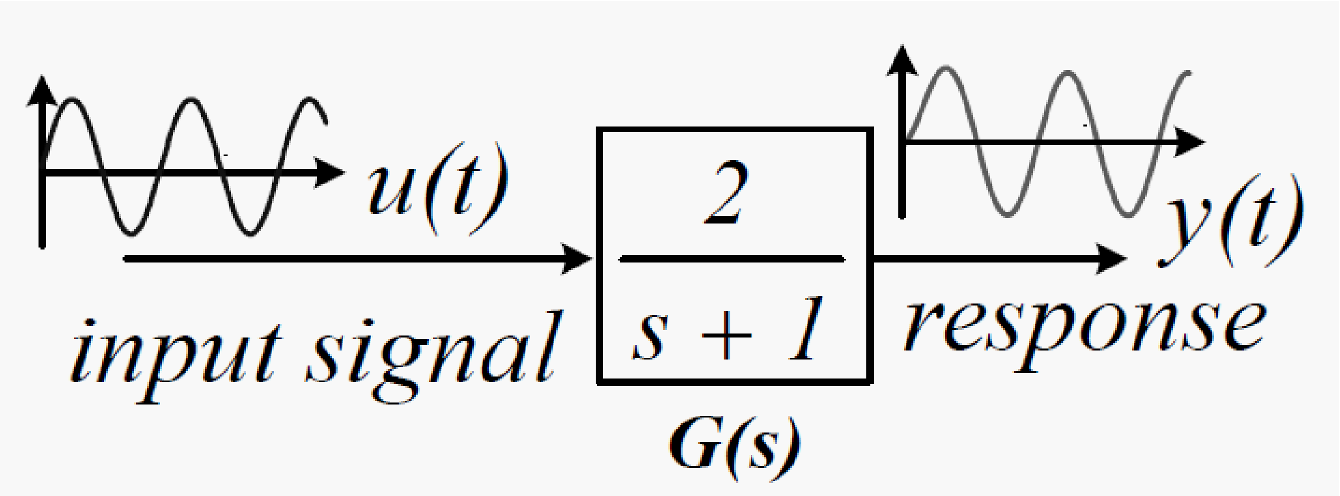


Best test signal would be sinusoidal

$$u(t) = X \sin \omega t$$



System response to a sinusoidal signal, 1



- Input: $u(t) = \sin t\,; \quad U(s) = \dfrac{1}{s^2+1}$
- Output:

$$Y(s) = G(s)U(s) = \frac{2}{(s+1)(s^2+1)} = \frac{s^2+1+1-s^2}{(s+1)(s^2+1)}$$

$$= \frac{1}{s+1} - \frac{s-1}{s^2+1} = \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

System response to a sinusoidal signal, 1

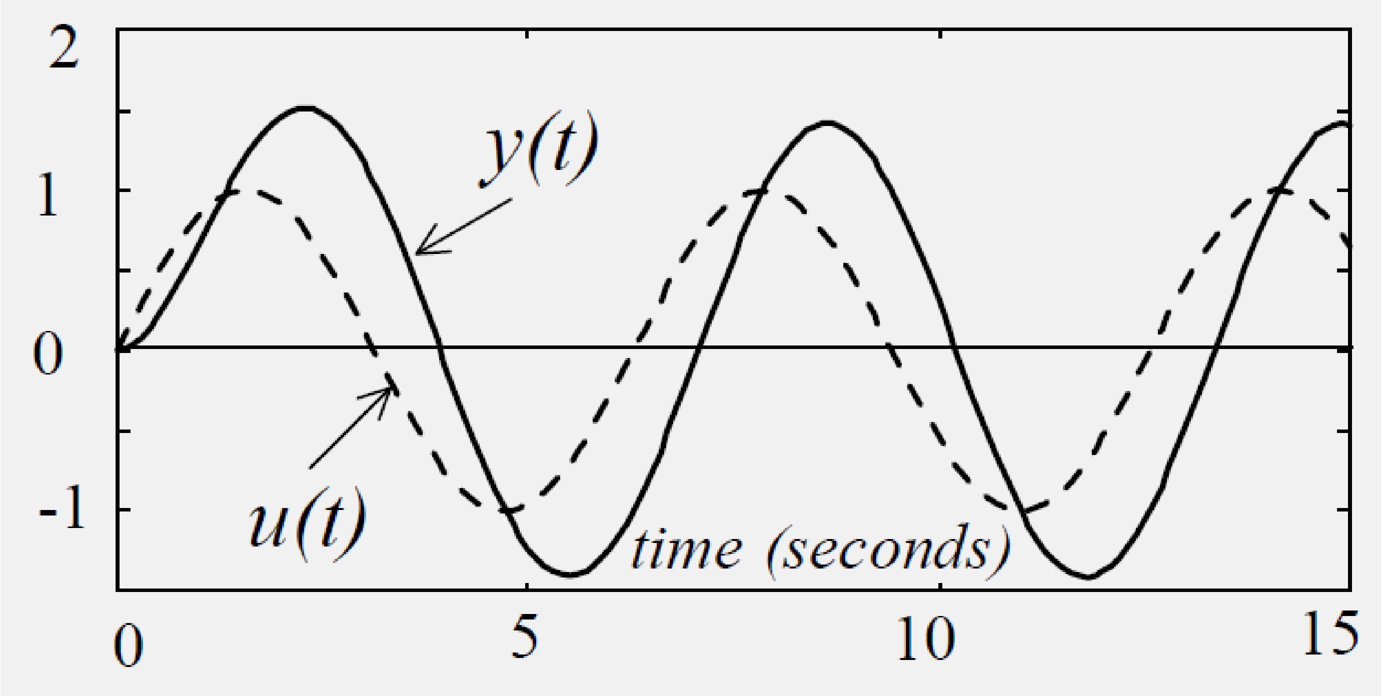
- Output: $Y(s) = \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$

$$
\begin{aligned}
y(t) = \mathcal{L}^{-1}\left[Y(s)\right] &= e^{-t} - \cos t + \sin t \\
&= e^{-t} + \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin t - \frac{1}{\sqrt{2}}\cos t\right) \\
&= e^{-t} + \sqrt{2}(\cos 45^o \sin t - \sin 45^o \cos t) \\
&= \underbrace{e^{-t}}_{\text{transient response } y_o(t)} + \underbrace{\sqrt{2}\sin(t - 45^o)}_{\text{steady state response } y_s(t)}
\end{aligned}
$$

System response to a sinusoidal signal, 1

$$u(t) = \sin t \quad \Rightarrow \quad y(t) = e^{-t} \; + \; \sqrt{2}\sin(t - 45^{o})$$

$$\text{when } \; t \to \infty \;\; y_0(t) \to 0 \quad y(t) \to y_s(t) = \sqrt{2}\sin(t - 45^{o})$$



Input:

Sinusoidal signal

Output: Sinusoidal signal with the same frequency, different amplitude and shifted phase.

System response to a sinusoidal signal, 2

- Input: $u(t) = \sin 2t\ ; \quad U(s) = \dfrac{2}{s^2 + 2^2}$
- Output:

$$Y(s) = G(s)U(s) = \frac{4}{(s+1)(s^2+4)} = \frac{4}{5}\frac{s^2+4+1-s^2}{(s+1)(s^2+4)}$$

$$= \frac{4}{5}\left(\frac{1}{s+1} - \frac{s-1}{s^2+4}\right) = \frac{4}{5}\left(\frac{1}{s+1} - \frac{s}{s^2+4} + \frac{1}{s^2+4}\right)$$

$$= \frac{4}{5}\left(\frac{1}{s+1} - \frac{s}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+2^2}\right)$$

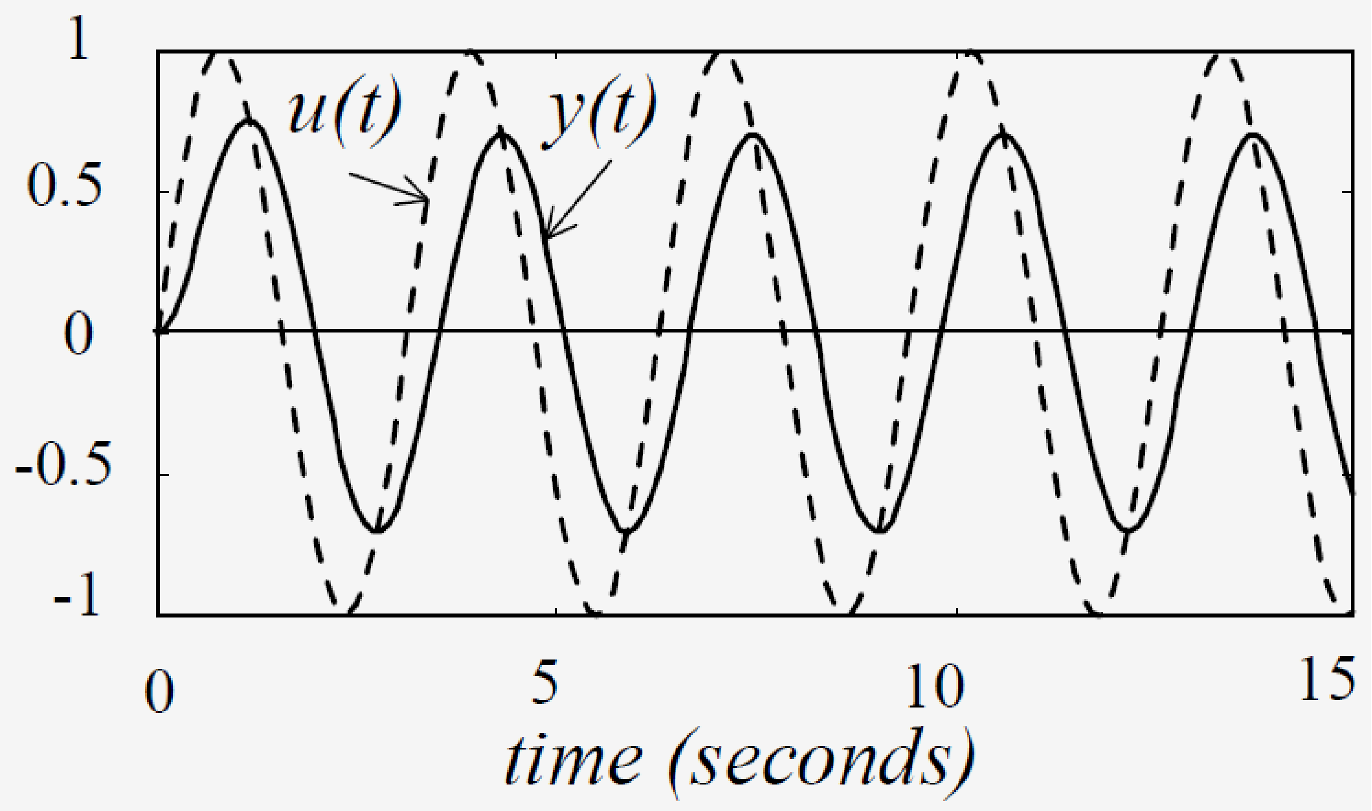
System response to a sinusoidal signal, 2

$$Y(s)=\frac{4}{5}\left(\frac{1}{s+1}-\frac{s}{s^2+2^2}+\frac{1}{2}\frac{2}{s^2+2^2}\right)$$

$$\begin{aligned}y(t)&=\frac{4}{5}e^{-t}-\frac{4}{5}\cos 2t+\frac{2}{5}\sin 2t\\&=\frac{4}{5}e^{-t}+\frac{2}{5}\sqrt{5}\left(\frac{1}{\sqrt{5}}\sin 2t-\frac{2}{\sqrt{5}}\cos 2t\right)\\&=\frac{4}{5}e^{-t}+\frac{2\sqrt{5}}{5}\left(\cos 63.4^o\sin 2t-\sin 63.4^o\cos 2t\right)\\&=0.8e^{-t}+0.89\sin(2t-63.4^o)\end{aligned}$$

System response to a sinusoidal signal, 2

$$u(t) = \sin 2t \Rightarrow y(t) \rightarrow 0.89 \sin(2t - 63.4^o)$$



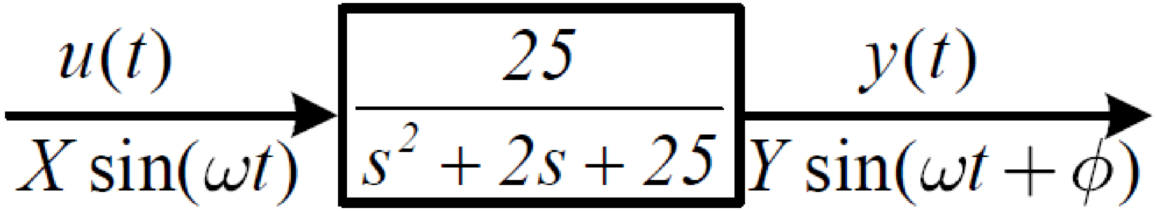
Input:

Sinusoidal signal

Output: Sinusoidal signal with the same frequency, different amplitude and shifted phase.

Responses to sinusoidal signals:

Experimental Testing



$$M=\frac{Y}{X},\quad \phi$$

M: Amplitude ratio

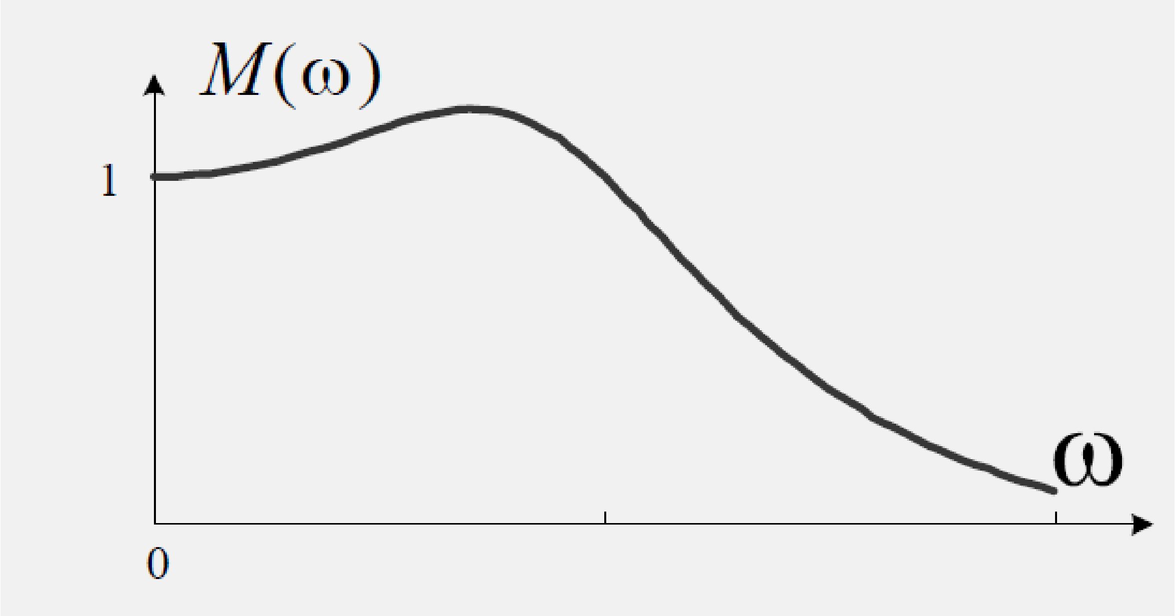
ϕ: Phase difference between $y(t)$ and $u(t)$.

Frequency response

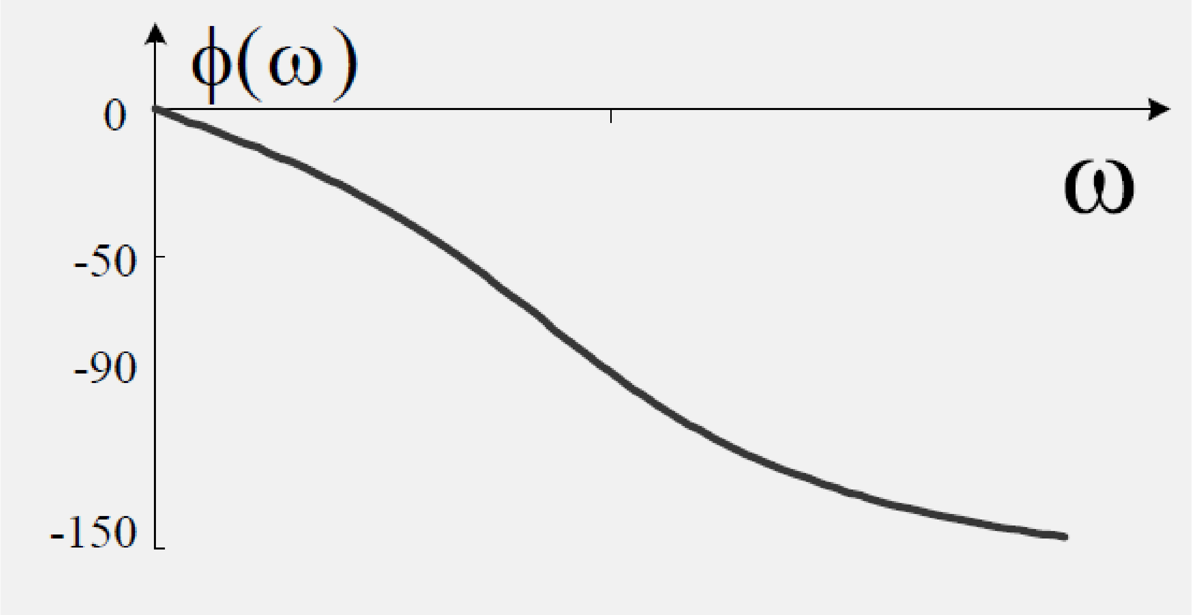
ω	$u(t)$	$y(t)$	M	ϕ
0	X	Y	1	0°
1	$X\sin(t)$	$1.04X\sin(t-4.76°)$	1.04	-4.76°
2	$X\sin(2t)$	$1.17X\sin(2t-10.8°)$	1.17	-10.8°
3	$X\sin(3t)$	$1.46X\sin(3t-20.6°)$	1.46	-20.6°
6	$X\sin(6t)$	$1.54X\sin(6t-132°)$	1.54	-132°
7	$X\sin(7t)$	$0.90X\sin(7t-150°)$	0.90	-150°
12	$X\sin(12t)$	$0.21X\sin(12t-168°)$	0.21	-168°

Frequency Response (FR)

$$M = \frac{Y}{X}, \quad \phi$$

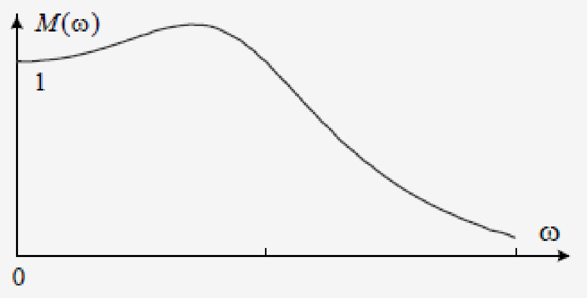


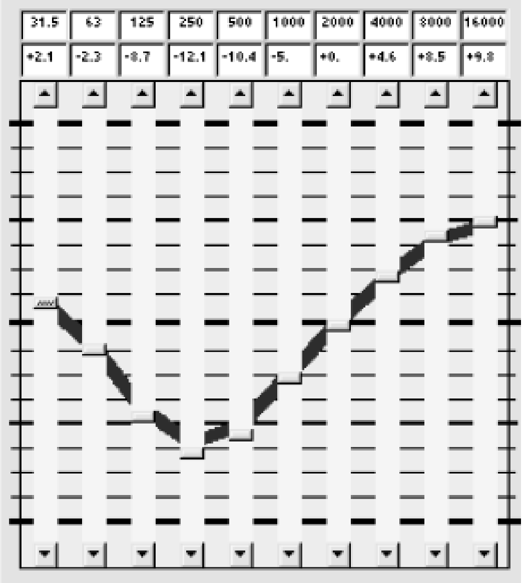
M: Output and input amplitude ratio



ϕ: Phase difference between output and input.

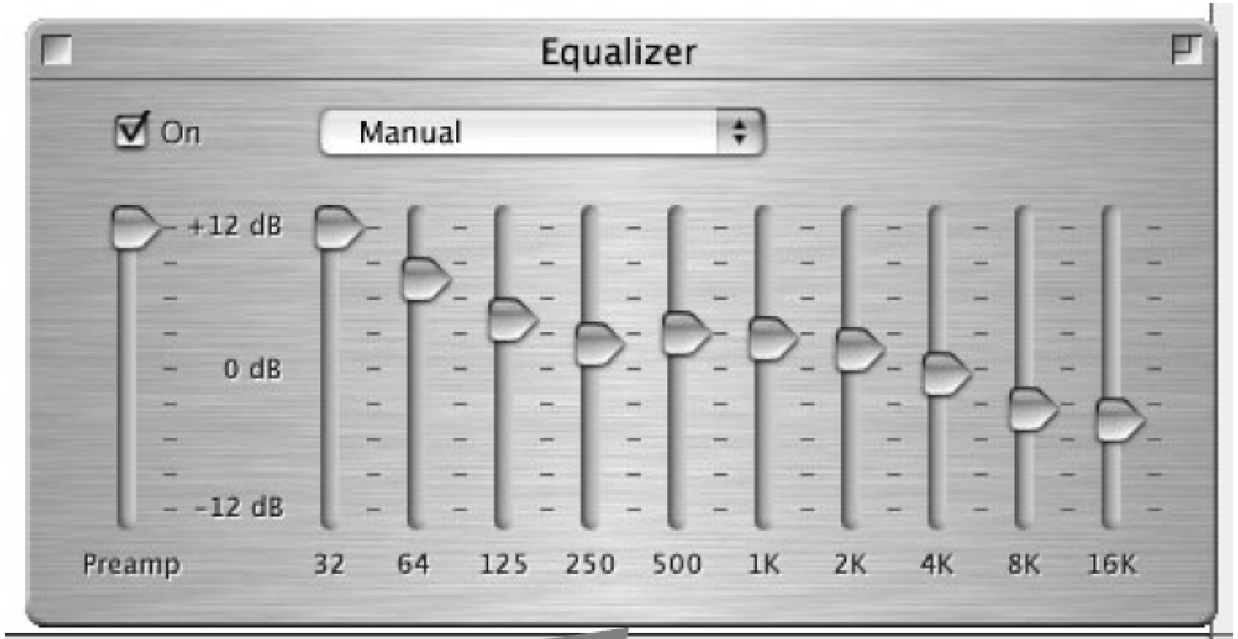
Where do you see FR in real life?
Hi-Fi/mp3 player





31.5	63	125	250	500	1000	2000	4000	8000	16000
+2.1	-2.3	-8.7	-12.1	-10.4	-5.	+0.	+4.6	+8.5	+9.8

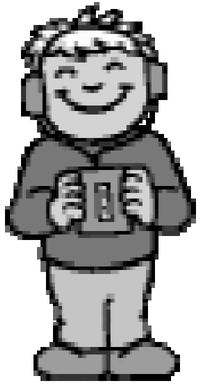
Graphic Equalizer



Equalizer

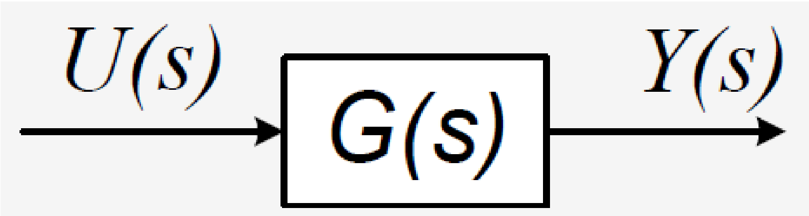
☑ On

Manual



Response to sinusoidal signal:

Theoretical Determination



$$u(t) = X \sin \omega t$$

$$U(s) = \frac{X\omega}{s^2 + \omega^2}$$

$$G(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$\boxed{\begin{aligned} s^2 + \omega^2 &= \\ &(s+j\omega)(s-j\omega) \end{aligned}}$$

$$Y(s) = G(s)U(s) = G(s)\frac{X\omega}{(s^2+\omega^2)}$$

$$= \frac{X\omega N(s)}{(s+j\omega)(s-j\omega)(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{A}{s+j\omega} + \frac{\overline{A}}{s-j\omega} + \frac{B_1}{s+p_1} + \cdots + \frac{B_n}{s+p_n}$$

Response to sinusoidal signal

$$Y(s) = \frac{A}{s + j\omega} + \frac{\bar{A}}{s - j\omega} + \frac{B_1}{s + p_1} + \cdots + \frac{B_n}{s + p_n}$$

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = \underbrace{Ae^{-j\omega t} + \bar{A}e^{j\omega t}}_{\text{steady state response } y_s(t)} + \underbrace{B_1 e^{-p_1 t} + \cdots + B_n e^{-p_n t}}_{\text{transient response } y_o(t)}$$

For stable systems: $y_0(t) \to 0$ when $t \to \infty$

$$y(t) \to y_s(t) = Ae^{-j\omega t} + \bar{A}e^{j\omega t} \quad \text{when } t \to \infty$$

Response to sinusoidal signal

$y(t) \rightarrow Ae^{-j\omega t} + \overline{A}e^{j\omega t}$ We need to find A and $\overline{A}$

$$Y(s) = G(s)\frac{X\omega}{(s^2+\omega^2)}$$

$$= \frac{A}{s+j\omega} + \frac{\overline{A}}{s-j\omega} + \frac{B_1}{s+p_1} + \cdots + \frac{B_n}{s+p_n}$$

Multiply both sides by $(s+j\omega)$

$$G(s)\frac{X\omega}{(s+j\omega)(s-j\omega)}(s+j\omega)$$

$$= A + (s+j\omega)\left(\frac{\overline{A}}{s-j\omega} + \frac{B_1}{s+p_1} + \cdots + \frac{B_n}{s+p_n}\right)$$

Response to sinusoidal signal

$$G(s)\frac{X\omega}{(s+\cancel{j\omega})(s-j\omega)}(s+\cancel{j\omega})$$

$$= A + (s+j\omega)\left(\frac{\bar{A}}{s-j\omega} + \frac{B_1}{s+p_1} + \cdots + \frac{B_n}{s+p_n}\right)$$

Let: $s = -j\omega$

$$G(-j\omega)\frac{X\omega}{-j\omega - j\omega} = A$$

$$A = -\frac{XG(-j\omega)}{2j}$$

Similarly, we can prove: $\bar{A} = \frac{XG(j\omega)}{2j}$

Response to sinusoidal signal

Input: Sinusoidal signal

$X \sin \omega t$

Output: Sinusoidal signal with the same frequency, different amplitude and shifted phase.

$Y \sin(\omega t + \phi)$

Output/input amplitude ratio

$$M = \frac{Y}{X} = |G(j\omega)|$$

The phase difference between output and input:

$$\phi = \angle G(j\omega)$$

- The variation of the amplitude and phase with frequency is called the *frequency response* of the system.

Frequency Responses

$$G(s) \xrightarrow[s=j\omega]{} G(j\omega)$$

- $G(j\omega)$, obtained by replacing s by $j\omega$ in the transfer function $G(s)$, is called as the *frequency response* of a system.

$$M(\omega) = \frac{Y}{X} = \left|G(j\omega)\right|$$

- *Gain frequency response*: the ratio of output amplitude to the input amplitude.

$$\phi(\omega) = \angle G(j\omega)$$

- *Phase frequency response*: the phase shift of the output relative to the input.

Gain and Phase Frequency Responses

$$G(j\omega) = \mathrm{Re}\{G(j\omega)\} + j\,\mathrm{Im}\{G(j\omega)\}$$

$$M(\omega) = \left|G(j\omega)\right|$$

- $|G(j\omega)|$ is the magnitude of $G(j\omega)$

$$\left|G(j\omega)\right| = \sqrt{\left(\mathrm{Re}\{\ G(j\omega)\}\right)^2 + \left(\mathrm{Im}\{\ G(j\omega)\}\right)^2}$$

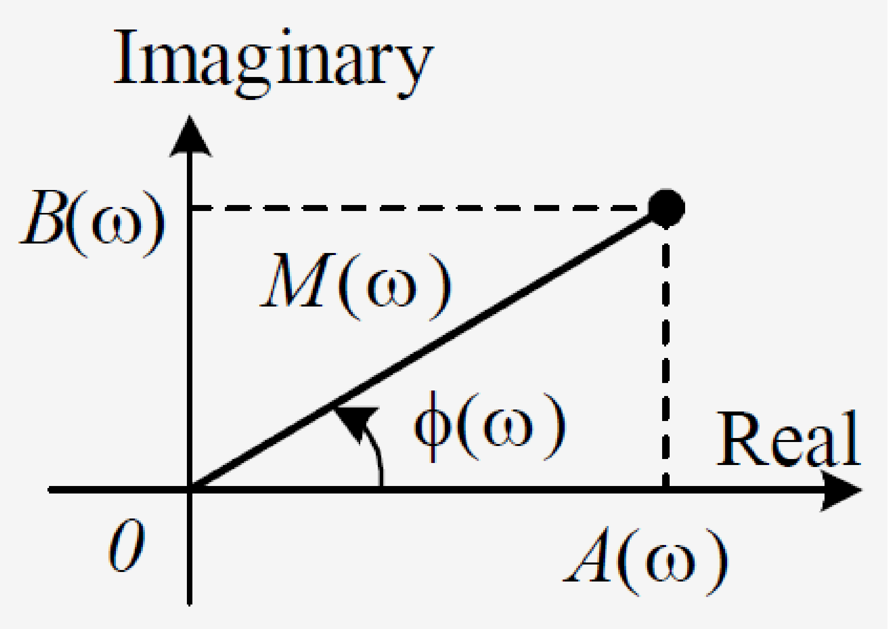
$$\phi(\omega) = \angle G(j\omega)$$

- ϕ represents the angle of $G(j\omega)$

$$\phi(\omega) = \angle G(j\omega) = \tan^{-1}\left[\frac{\mathrm{Im}\{\ G(j\omega)\}}{\mathrm{Re}\{\ G(j\omega)\}}\right]$$

Gain and Phase: Case I

$$G(j\omega) = A(\omega) + jB(\omega)$$



- Gain: $M(\omega) = \left|G(j\omega)\right| = \sqrt{A(\omega)^2 + B(\omega)^2}$
- Phase: $\phi(\omega) = \angle G(j\omega) = \tan^{-1}\dfrac{B(\omega)}{A(\omega)}$

Gain and Phase: Case II

$$G(j\omega) = \frac{A(\omega) + jB(\omega)}{C(\omega) + jD(\omega)}$$

- Gain:

$$M(\omega) = \left|G(j\omega)\right| = \frac{\left|A(\omega) + jB(\omega)\right|}{\left|C(\omega) + jD(\omega)\right|} = \frac{\sqrt{A(\omega)^2 + B(\omega)^2}}{\sqrt{C(\omega)^2 + D(\omega)^2}}$$

- Phase:

$$\phi(\omega) = \angle G(j\omega) = \angle\left[A(\omega) + jB(\omega)\right] - \angle\left[C(\omega) + jD(\omega)\right]$$

$$= \tan^{-1}\frac{B(\omega)}{A(\omega)} - \tan^{-1}\frac{D(\omega)}{C(\omega)}$$

Gain and Phase: Case III

$$G(j\omega) = \frac{1}{C(\omega) + jD(\omega)}$$

- Gain:

$$M(\omega) = \left|G(j\omega)\right| = \frac{|1|}{\left|C(\omega) + jD(\omega)\right|} = \frac{1}{\sqrt{C(\omega)^2 + D(\omega)^2}}$$

- Phase:

$$\phi(\omega) = \angle G(j\omega) = \angle[1] - \angle[C(\omega) + jD(\omega)]$$

$$= -\tan^{-1}\frac{D(\omega)}{C(\omega)}$$

Frequency Response Example, 1

$$G(s)=\frac{2}{s+1}\xrightarrow{\text{Replacing } s \text{ with } j\omega} G(j\omega)=\frac{2}{j\omega+1}$$

$$M(\omega)=|G(j\omega)|=\left|\frac{2}{1+j\omega}\right|=\frac{|2|}{|1+j\omega|}=\frac{2}{\sqrt{1^2+\omega^2}}$$

$$\phi(\omega)=\angle G(j\omega)=\angle\left(\frac{2}{1+j\omega}\right)$$

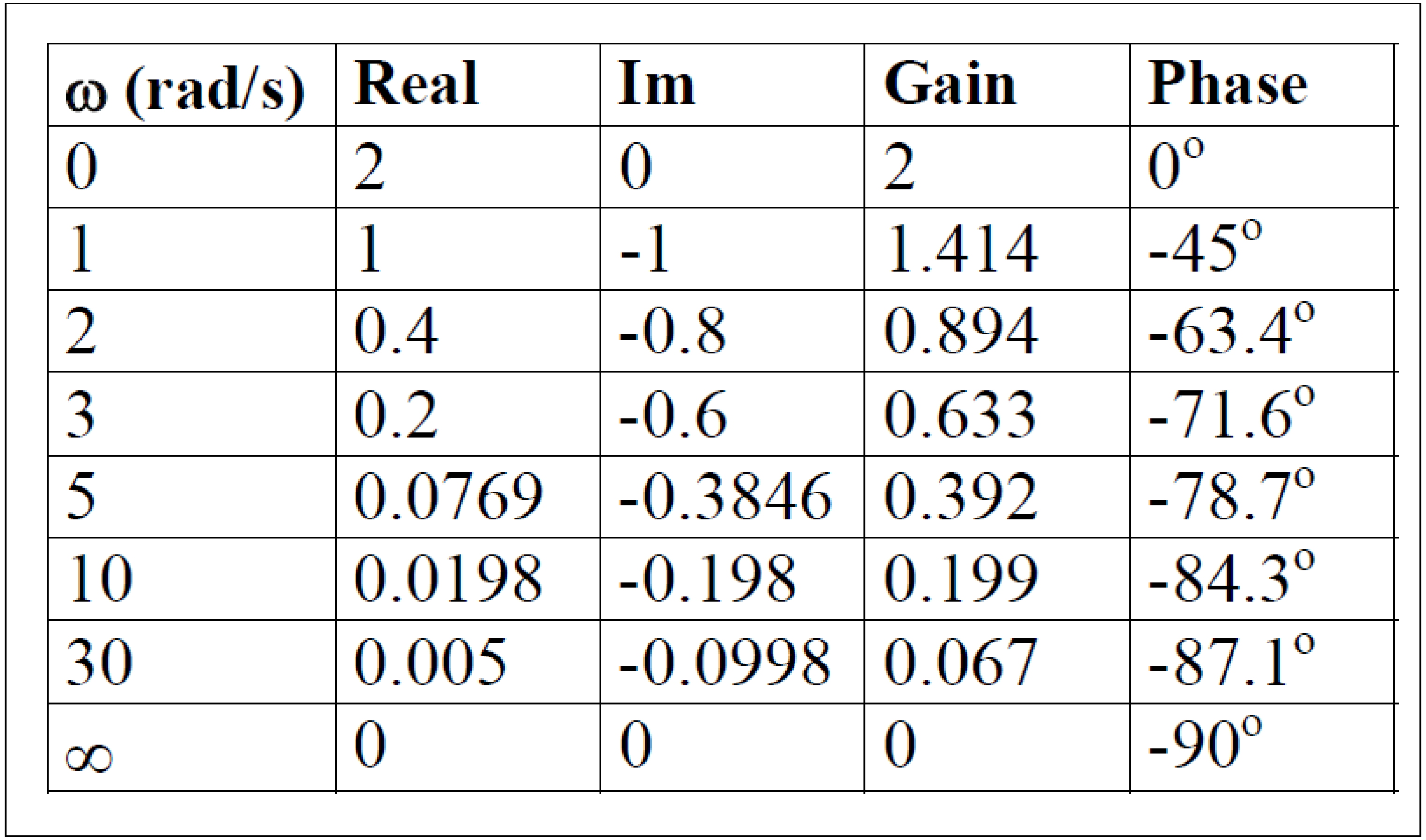
$$=\angle(2)-\angle(1+j\omega)=0-\tan^{-1}(\frac{\omega}{1})$$

$$=-\tan^{-1}\omega$$

Frequency Response Example, 1

$$M(\omega) = \frac{2}{\sqrt{1+\omega^2}} \qquad \phi(\omega) = -\tan^{-1}\omega$$

ω (rad/s): nature (angular) frequency, normally called as frequency



ω (rad/s)	Real	Im	Gain	Phase
0	2	0	2	0°
1	1	-1	1.414	-45°
2	0.4	-0.8	0.894	-63.4°
3	0.2	-0.6	0.633	-71.6°
5	0.0769	-0.3846	0.392	-78.7°
10	0.0198	-0.198	0.199	-84.3°
30	0.005	-0.0998	0.067	-87.1°
∞	0	0	0	-90°