Instrumentation and Controls
ETM 3301

Lecture 19

Instructor
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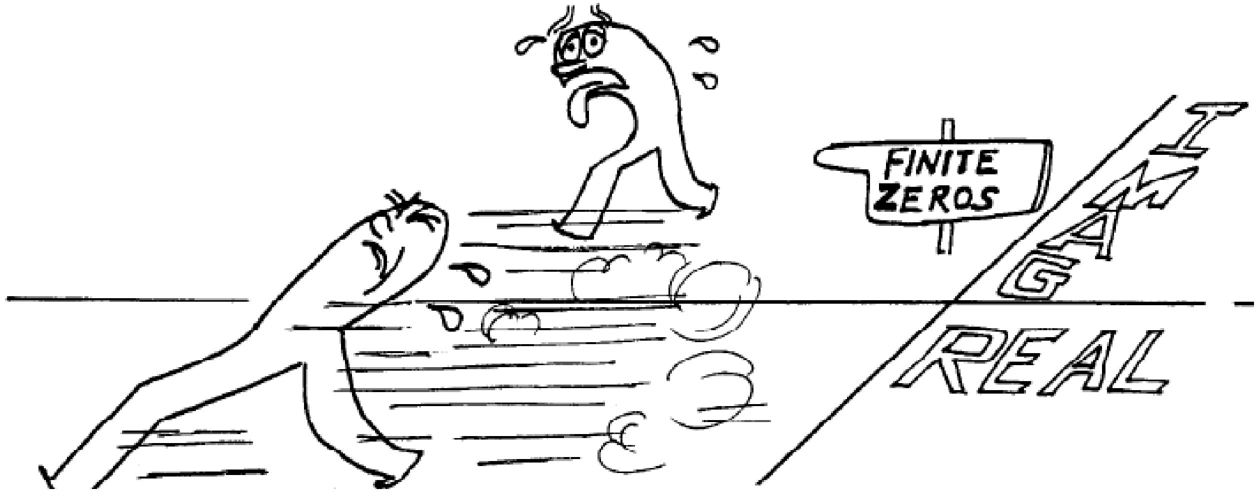
Root Locus Property: Symmetry

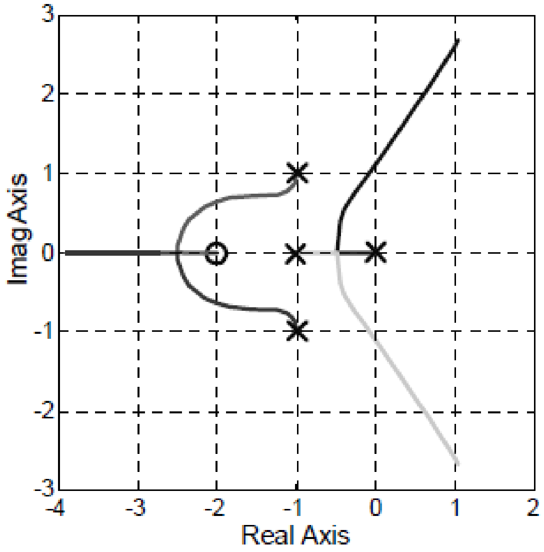
- *The root locus plot is symmetric about the real axis.*
  - This is because complex roots occur in complex conjugate pairs.

$$s^2 + 2s + K = 0$$

$$s_{1,2} = -1 \pm j\sqrt{K-1}$$



"WHOA..WHOA...take it easy, man! Don't you know we'are supposed to be conjugates and get there asymptotically?"



Root Locus Property – Number of branches

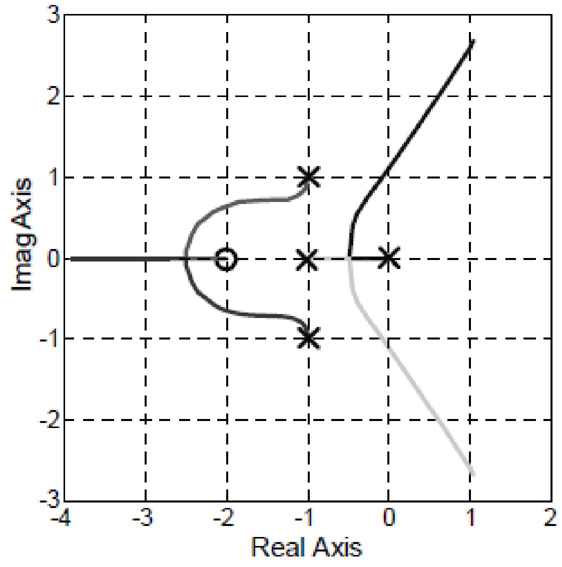
- *The number of branches of the root locus equals the number of closed-loop poles and also equals to the number of open-loop poles.*

$$KP(s) = K\frac{N(s)}{D(s)} \qquad \begin{matrix} N(s)\ order\ m \\ D(s)\ order\ n \end{matrix} \qquad n \geq m$$

$$CLCE: \quad D(s) + KN(s) = 0 \qquad order\ \ n$$

The number of closed-loop poles equals to the number of open-loop poles, n.

$$K\frac{(s+2)}{s(s+1)(s^2+2s+2)}$$



Root Locus Property - Starting and ending points

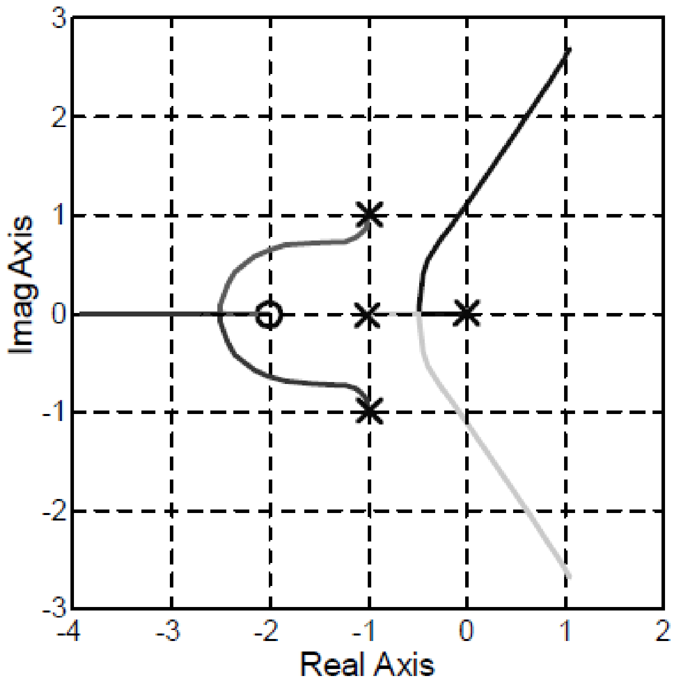
- *The loci start from the open-loop poles with K=0 and terminate with K=+∞ either at the open-loop zeros or at infinity.*

$$K\frac{(s+2)}{s(s+1)(s^2+2s+2)}$$

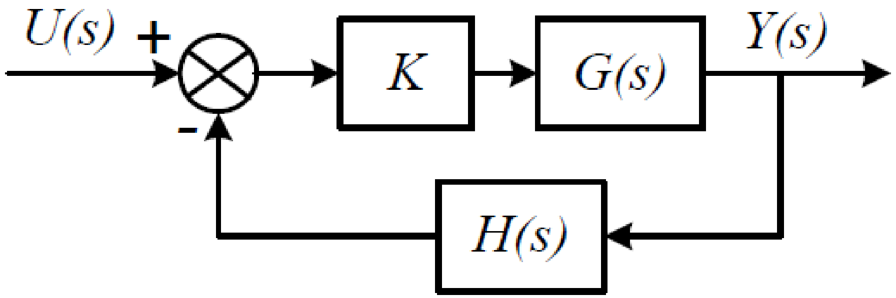
Zero: *-2*

Poles: *0, -1*

-1±j



Open-loop, Closed Loop Poles and Zero Relations



$m_f \leq n_f$

$m_h \leq n_h$

Forward path TF: $KG(s) = K\dfrac{N_f(s)}{D_f(s)}$ ← Order m_f (numerator), Order n_f (denominator)

Feedback TF: $H(s) = \dfrac{N_h(s)}{D_h(s)}$ ← Order m_h (numerator), Order n_h (denominator)

Closed-loop TF:

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)} = \frac{KN_f(s)D_h(s)}{D_h(s)D_f(s)+KN_f(s)N_h(s)}$$

Numerator: Order $m_f + n_h$

Denominator: Order $n = n_f + n_h$

Open-loop, Closed Loop Poles and Zero Relations

Closed-loop TF: $$T(s)=\frac{KG(s)}{1+KG(s)H(s)}=\frac{KN_f(s)D_h(s)}{D_h(s)D_f(s)+KN_f(s)N_h(s)}$$

Closed-loop system zeros: Solutions of $KN_f(s)D_h(s)=0$

Solutions are forward path TF zeros → $N_f(s)=0$

Solutions are feedback TF poles → $D_h(s)=0$

Closed-loop TF zeros = {forward path TF zeros}

+ {feedback TF poles}

Open-loop, Closed Loop Poles and Zero Relations

Closed-loop TF: $$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{KN_f(s)D_h(s)}{D_h(s)D_f(s) + KN_f(s)N_h(s)}$$

Open-loop TF: $$KP(s) = KG(s)H(s) = K\frac{N_f(s)N_h(s)}{D_f(s)D_h(s)} = K\frac{N(s)}{D(s)}$$

Closed-loop characteristic equation (CLCE):

$$1 + KG(s)H(s) = 0 \quad \Rightarrow \quad D(s) + KN(s) = 0$$

Closed-loop poles depend on the value of K!!!

Open-loop, Closed Loop Poles and Zero Relations

Closed-loop TF:

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

Open-loop TF:

$$KP(s) = K\frac{N(s)}{D(s)}$$

Closed-loop characteristic equation (CLCE):

$$1 + KG(s)H(s) = 0 \quad \Rightarrow \quad D(s) + KN(s) = 0$$

CL poles: solutions → $D(s) + KN(s) = 0$

K=0

$$D(s) = 0 \xleftarrow{K=0} D(s) + KN(s) = 0$$

Solutions are open loop poles

When K= 0, Closed-loop TF Poles = Open-loop TF poles

Open-loop, Closed Loop Poles and Zero Relations

Closed-loop TF:

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}$$

Open-loop TF:

$$KP(s)=K\frac{N(s)}{D(s)}$$

Closed-loop characteristic equation (CLCE):

$$1+KG(s)H(s)=0 \quad \Rightarrow \quad D(s)+KN(s)=0$$

CL poles: solutions → $D(s)+KN(s)=0$

K=∞

$$N(s)=0 \xleftarrow{K=\infty} \frac{1}{K}D(s)+N(s)=0$$

Solutions are open loop zero

When K= ∞, some closed-loop TF Poles

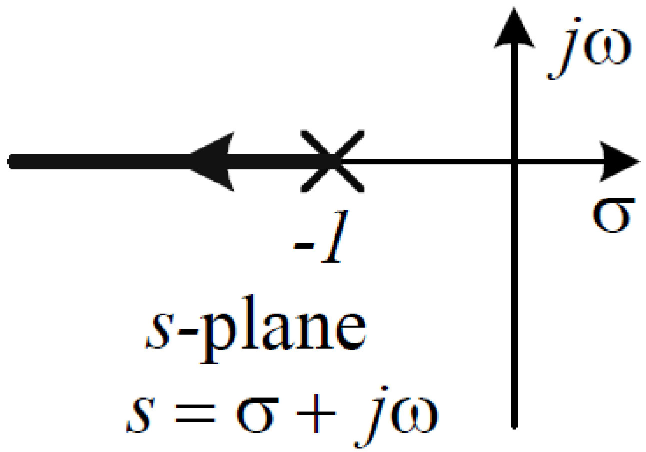
= Open-loop TF zeros

Root Locus Property

- Real-axis segments
  - *A root locus exists on the real axis in any section for which the total number of poles and zeros of P(s) to the right is <u>odd.</u>*

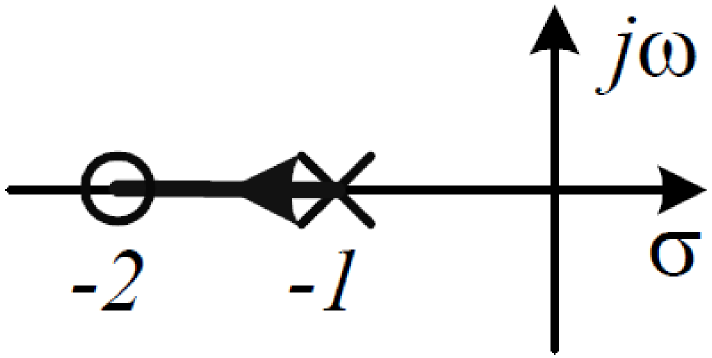
$$KP(s) = K\frac{1}{(s+1)}$$

OL pole: -1



Root Locus Property

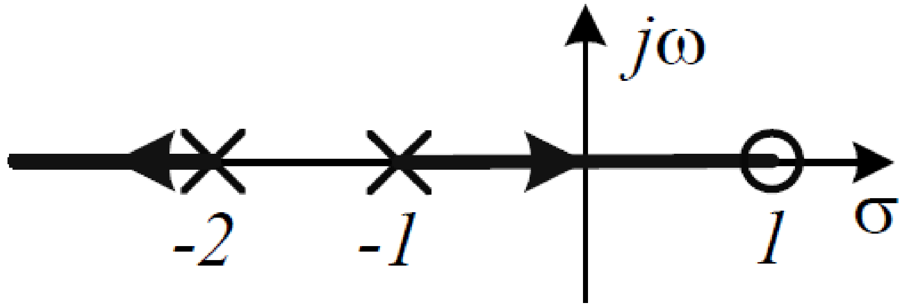
$$KP(s) = K\frac{(s+2)}{(s+1)}$$



OL pole: -1

OL zero: -2

$$K\frac{(s-1)}{(s+1)(s+2)}$$



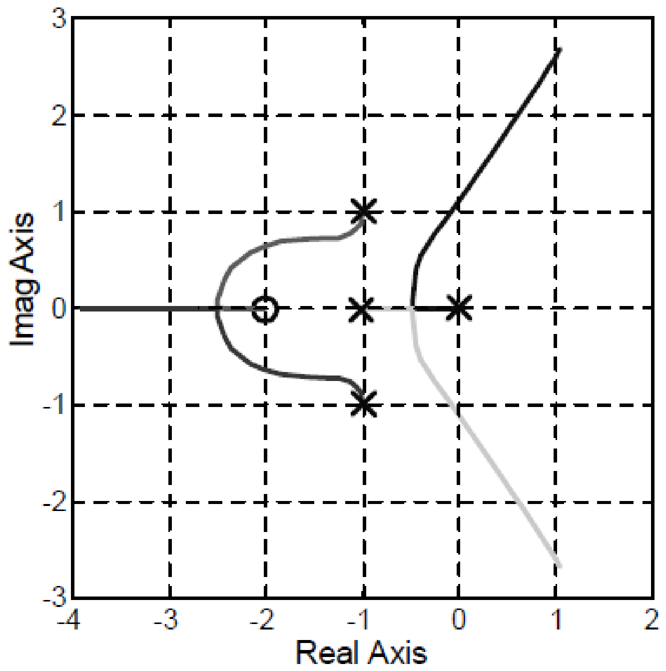
OL poles: -1, -2

OL zero: 1

Root Locus Property

- Behaviour at infinity
  - *As K→∞, m locus branches approach OL zeros and n-m branches approach infinity. (n: OL pole number; m: OL zero number).*

- *The root locus approaches straight lines as asymptotes as the locus approaches infinity.*



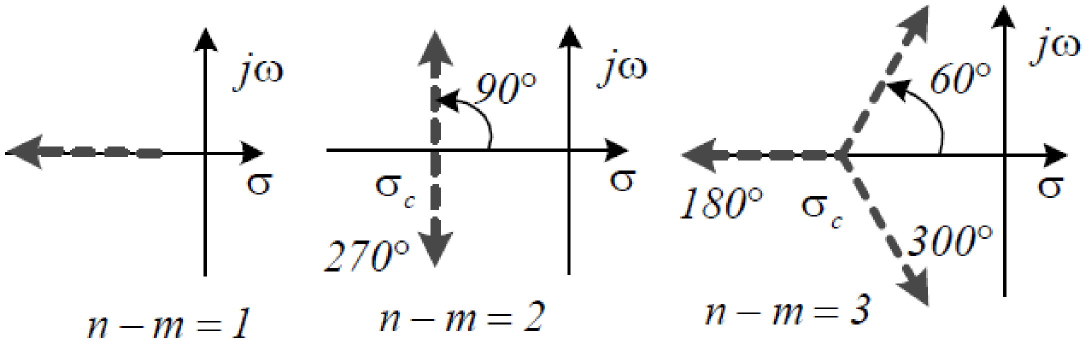
Root Locus Property

- *The angles that asymptotes make with the real axis are determined as:*

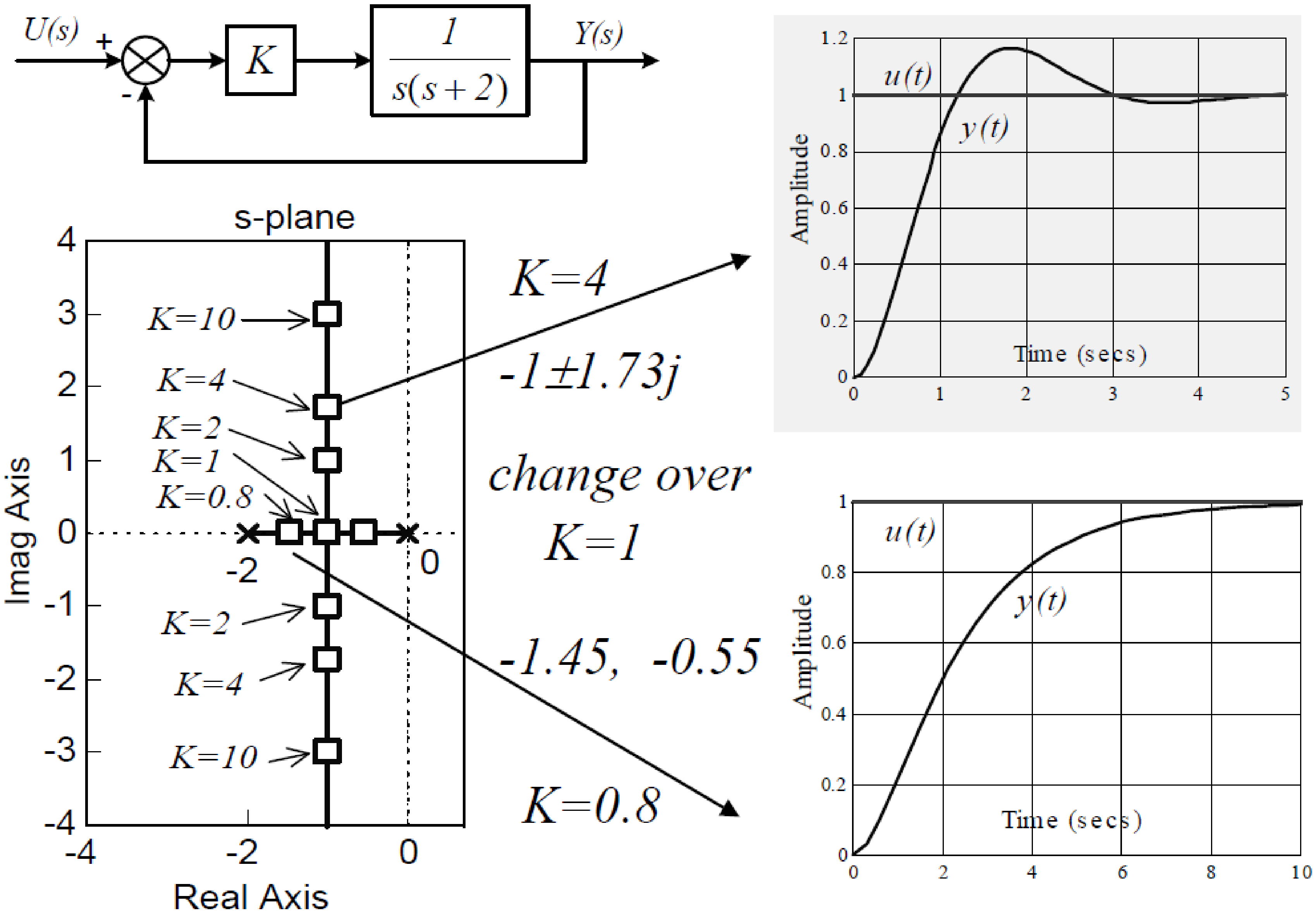
$$\alpha_k = \frac{180° + k \cdot 360°}{n-m}, \quad k = 0, 1, 2, \cdots, (n-m-1)$$

- *The asymptotes intersect the real axis at the common point*

$$\sigma_c = \frac{\sum poles\ of\ P(s) - \sum zeros\ of\ P(s)}{n-m} \quad n-m \geq 2$$



Real and Complex Poles: Transition point

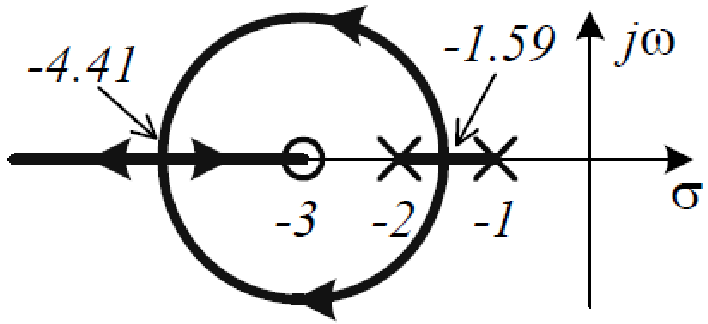


Real and Complex Poles: Transition point

- Real-axis breakaway & break-in points
  - *The real axis breakaway and break-in points can be found by solving the equation:*

$$\frac{dK}{ds} = \frac{d}{ds}\left(\frac{-1}{P(s)}\right) = 0$$

where $1 + KP(s) = 0 \quad \Rightarrow \quad K = -\frac{1}{P(s)}$



Real and Complex Poles: Transition point

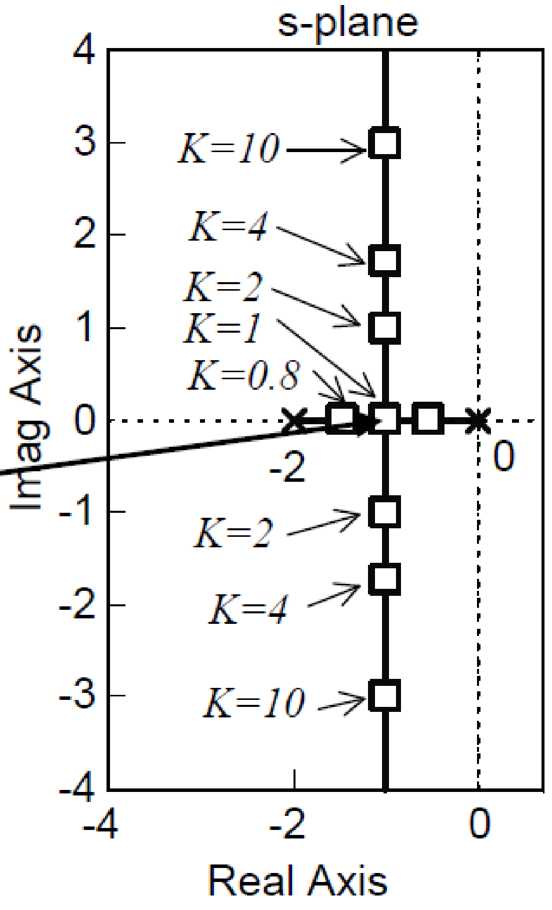
Example: Open-loop TF $KP(s) = K\dfrac{1}{s(s+2)}$

$$K = -\frac{1}{P(s)} = -s(s+2) = -s^2 - 2s$$

$$\frac{dK}{ds} = \frac{d}{ds}\left(-s^2 - 2s\right) = -2s - 2 = 0$$

$s = -1$ (breakaway)

Corresponding value for K $K = -(1)^2 - 2(1) = 1$



Real and Complex Poles: Transition point

Example:

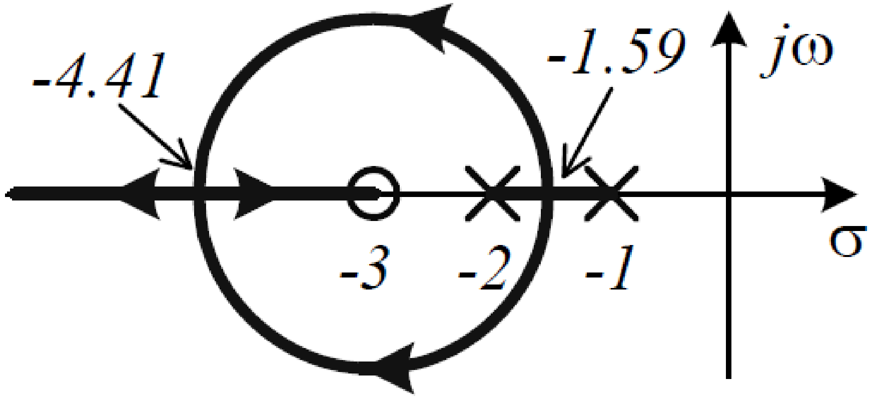
$$KP(s) = K\frac{(s+3)}{(s+1)(s+2)}$$

$$K = -\frac{1}{P(s)} = -\frac{(s+1)(s+2)}{s+3}$$

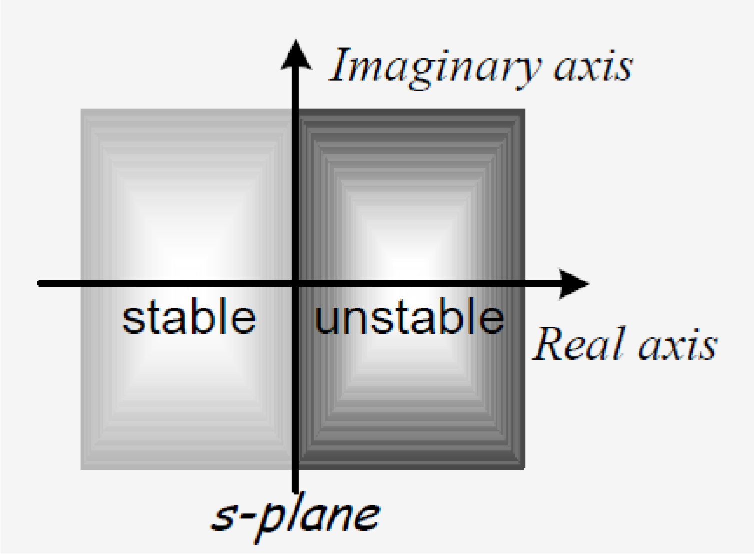
$$\frac{dK}{ds} = \frac{d}{ds}\left(-\frac{(s+1)(s+2)}{s+3}\right) = -\frac{s^2+6s+7}{(s+3)^2} = 0$$

$s_1 = -1.59$ (breakaway)

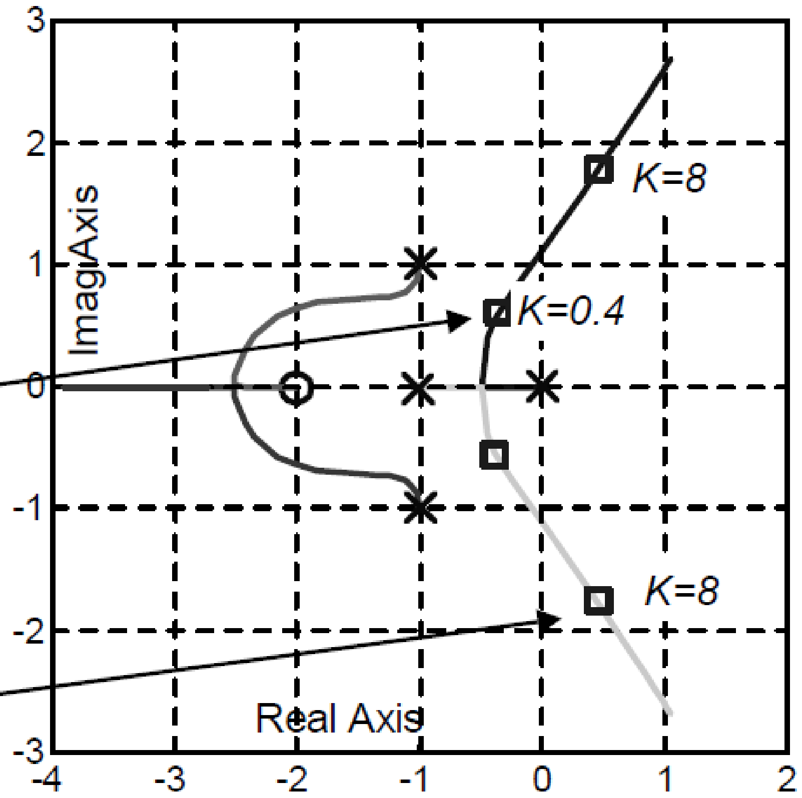
$s_2 = -4.41$ (break-in)



Separation between stable region and unstable region



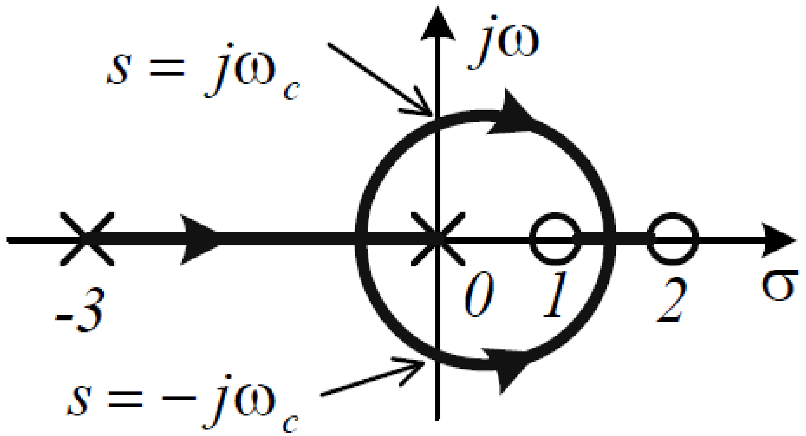
$$\frac{K(s+2)}{s(s+1)(s^2+2s+2)}$$



$-1.11 \pm 0.77j$ $-0.39 \pm 0.53j$	$K=0.4$
$0.5 \pm 1.83j$ $-2.0 \pm 0.64j$	$K=8$

- # Imaginary axis crossing
  - *The points at which the loci cross the imaginary axis and the associated values of K can be found by substituting s=jω into the closed-loop characteristic equation.*

Example $$G(s) = KP(s) = \frac{K(s-1)(s-2)}{s(s+3)}$$



Determine Stability Region via Imaginary axis crossing

CLCE $$1+\frac{K(s-1)(s-2)}{s(s+3)}=0 \Rightarrow s(s+3)+K(s-1)(s-2)=0$$

Rearrange: $$(1+K)s^2+3(1-K)s+2K=0$$

When the root locus crosses the imaginary axis, the root (closed-loop pole) is *pure* imaginary. Let: $s=j\omega_c$

$$(1+K)(j\omega_c)^2+3(1-K)j\omega_c+2K=0$$
$$\Rightarrow -(1+K)\omega_c^2+3(1-K)j\omega_c+2K=0$$

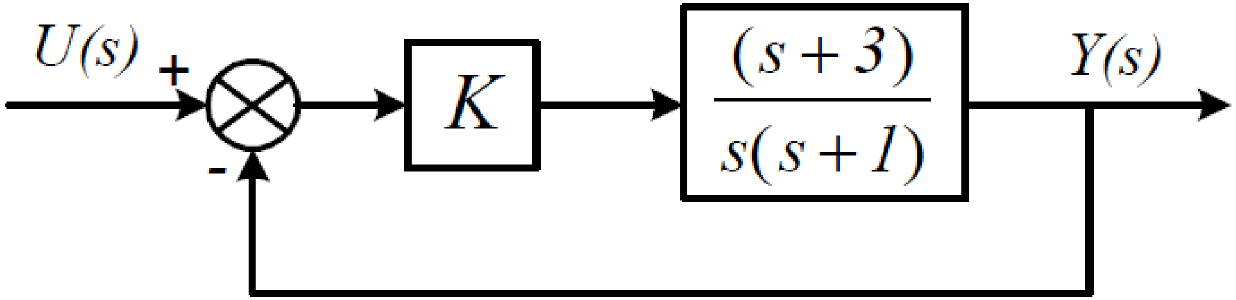
Both real and imaginary parts must be zero.

$$3(1-K)\omega_c=0 \quad \Rightarrow \quad K=1$$
$$-(1+K)\omega_c^2+2K=0 \quad \Rightarrow \omega_c=1$$

when $K>1$, the system becomes unstable

Root Locus Design Example

- Find the parameter K which makes the system with a damping ratio of $\zeta=0.866$.



CLTF:

$$T(s)=\frac{K\dfrac{s+3}{s(s+1)}}{1+K\dfrac{s+3}{s(s+1)}\times 1}=\frac{K(s+3)}{s^2+(1+K)s+3K}$$

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

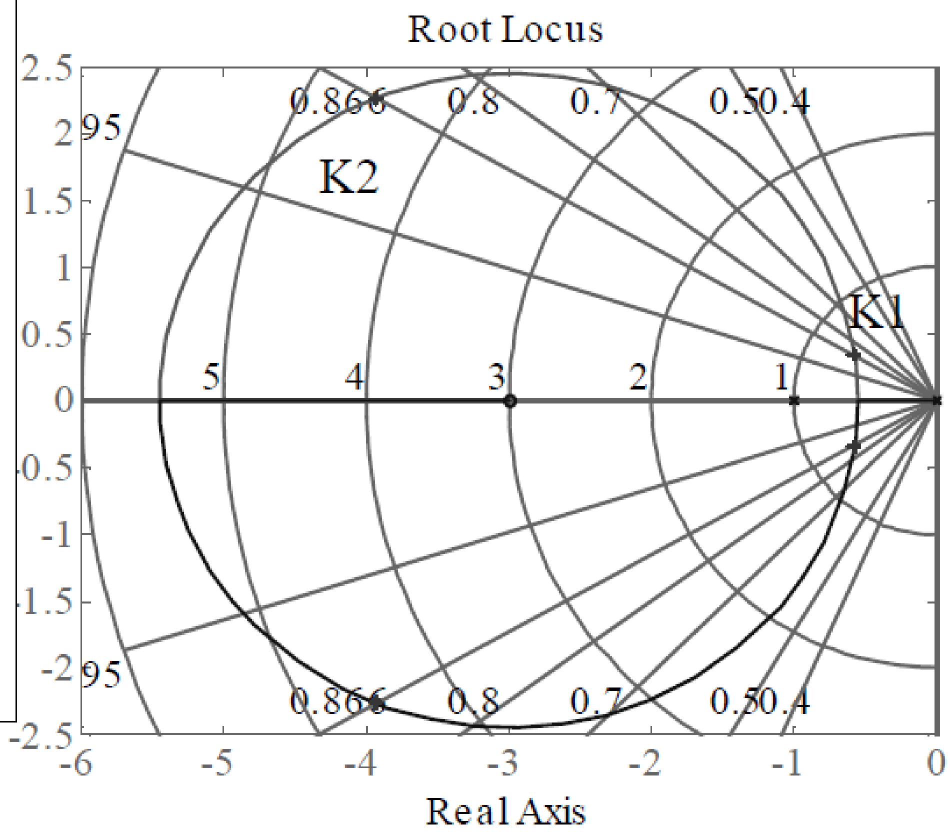
Cannot use second order system formula!

$$\zeta=0.866$$

$$PO=0.43\%$$

Root Locus Design Example: Matlab solution

```
G=tf([1 3], [1 1 0])
rlocus(G)
z=[0.4 0.5 0.7 0.8 0.866 0.95]
w=[1 2 3 4 5 6]
sgrid(z,w)
[K1,poles1]=rlocfind(G)
[K2,poles2]=rlocfind(G)
T1=feedback(K1*G,1)
T2=feedback(K2*G,1)
ltiview
```



K1=0.1498

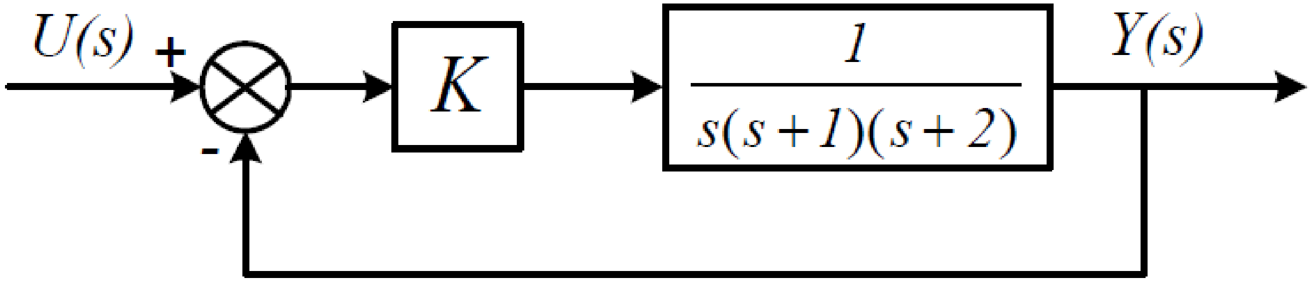
K2=6.8640

PO=0.44% when K=K1=0.1498

PO=10.5% when K=K2=6.8640

Root Locus Design Example

- Find the parameter K which makes the system with a damping ratio of ζ=0.5.



CLTF:

$$T(s) = \frac{\dfrac{K}{s(s+1)(s+2)}}{1 + \dfrac{K}{s(s+1)(s+2)} \times 1} = \frac{K}{s^3 + 3s^2 + 2s + K}$$

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

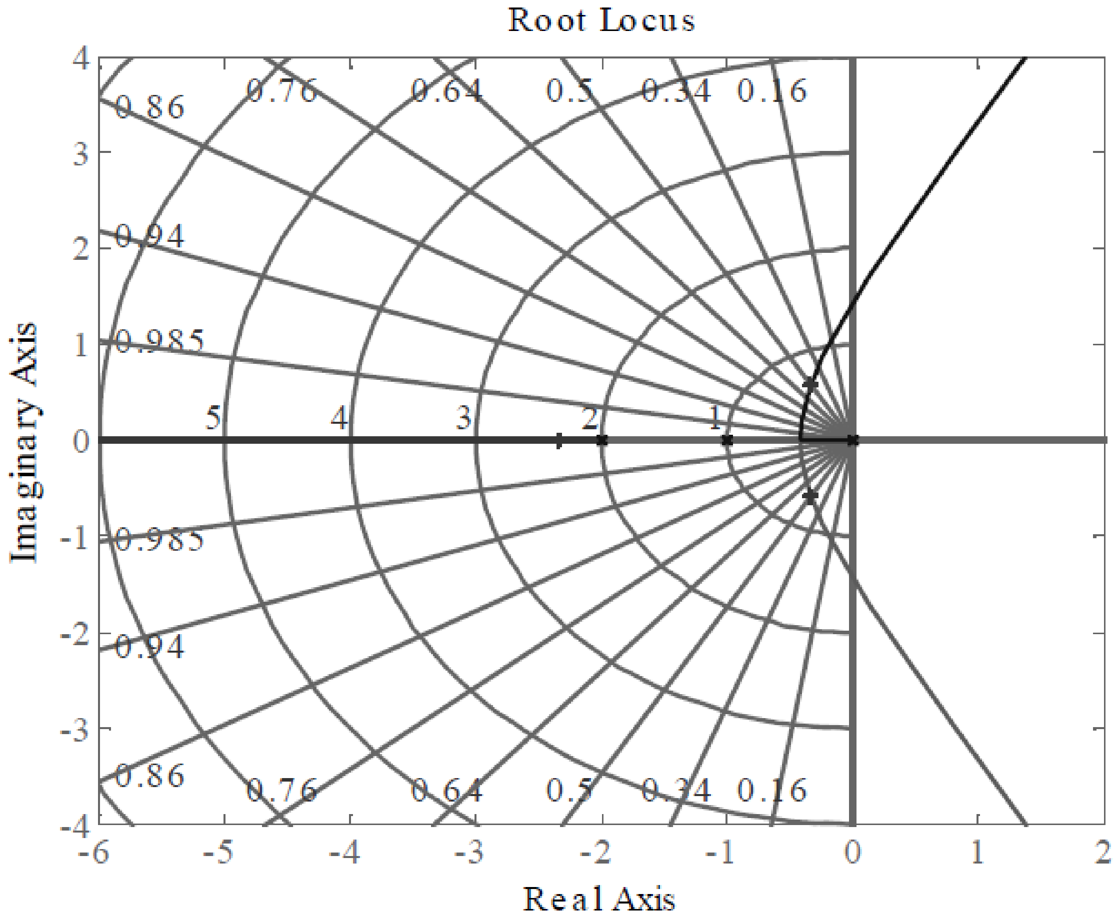
Cannot use second system formula!

$$\zeta = 0.5$$

$$PO = 16.3\%$$

Root Locus Design Example: Matlab solution

```
G=tf([1],[1 3 2 0])
rlocus(G)
sgrid
[K,poles]=rlocfind(G)
T=feedback(K*G,1)
ltiview
```

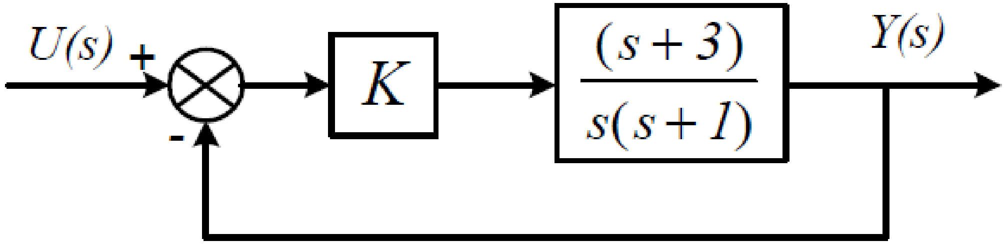


Results:

K=1.04

PO=15.6 % when K=1.04

Root Locus Design via Angle and Magnitude Conditions



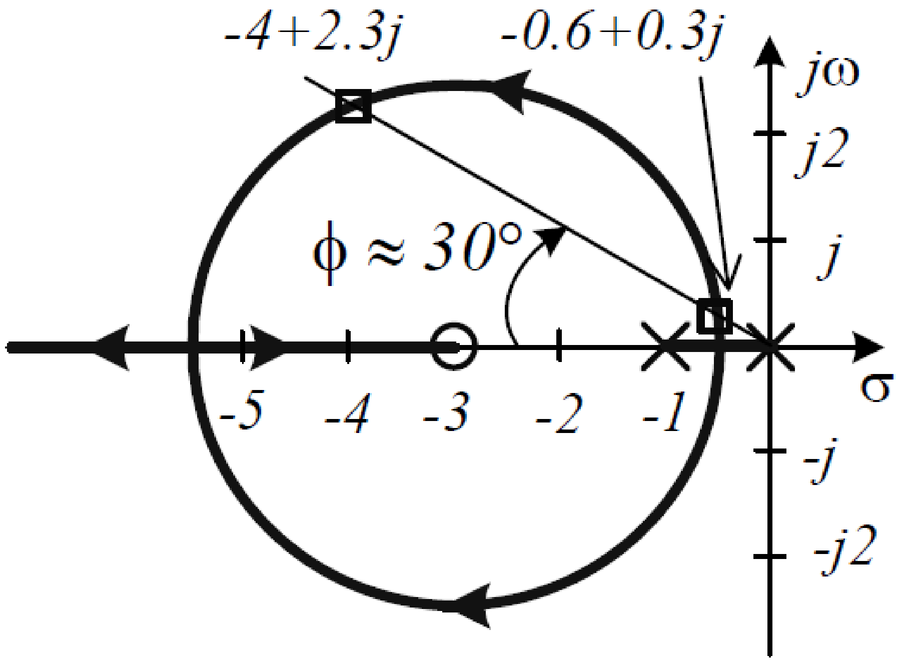
OL poles: *0, -1*

OL zeros: *-3*

Number of branches: 2, start from *0, -1*; one end at *-3*, one end at infinity

Root locus on the real axis sections:

[-∞ -3] [-1 0]



Root Locus Design via Angle and Magnitude Conditions

$$KP(s) = K\frac{(s+3)}{s(s+1)}$$

We need to find breakaway point

CLCE: $$1 + K\frac{(s+3)}{s(s+1)} = 0$$

$$K = -\frac{s(s+1)}{(s+3)}$$

$$\frac{dK}{ds} = -\frac{s^2+6s+3}{(s+3)^2} = 0$$

$$s_1 = -0.55 \quad s_2 = -5.45$$

breakaway point (s_1) break-in point (s_2)

Root Locus Design via Angle and Magnitude Conditions

$$\phi = \cos^{-1}\zeta = \cos^{-1} 0.866 \approx 30^\circ$$

Draw a constant damping ratio line on the *s*-plane, this line crosses the root locus on two locations (found from the graph): $s_1 \approx -0.6 + 0.3j$ $s_2 \approx -4 + 2.3j$

We can use the angle condition to check if these two points are actually on the root locus.

$$\angle P(s) = 180^\circ \pm k \cdot 360^\circ \quad ; \quad k = 0, 1, 2, \cdots$$

$$P(s) = \frac{(s+3)}{s(s+1)}$$

Root Locus Design via Angle and Magnitude Conditions

$$\angle P(s_1) = \angle\left\{\frac{s_1+3}{s_1(s_1+1)}\right\}$$

$$= \angle\{s_1+3\} - \angle s_1 - \angle\{s_1+1\}$$

$$= \angle\{2.4+0.3j\} - \angle\{-0.6+0.3j\} - \angle\{0.4+3j\}$$

$$= 7.13^\circ - 153.43^\circ - 36.87^\circ = -183.16^\circ \approx -180^\circ$$

s_1 is roughly on the root locus.

Root Locus Design via Angle and Magnitude Conditions

$$\angle P(s_2) = \angle\left\{\frac{s_2+3}{s_2(s_2+1)}\right\}$$

$$= \angle\{s_2+3\} - \angle s_2 - \angle\{s_2+1\}$$

$$= \angle\{-1+2.3j\} - \angle\{-4+2.3j\} - \angle\{-3+2.3j\}$$

$$= 113.5^\circ - 150.1^\circ - 142.5^\circ = -179.1^\circ \approx -180^\circ$$

s_2 is roughly on the root locus.

Root Locus Design via Angle and Magnitude Conditions

According to the magnitude condition:

$$|KP(s_1)| = K\left|\frac{s_1+3}{s_1(s_1+1)}\right|$$

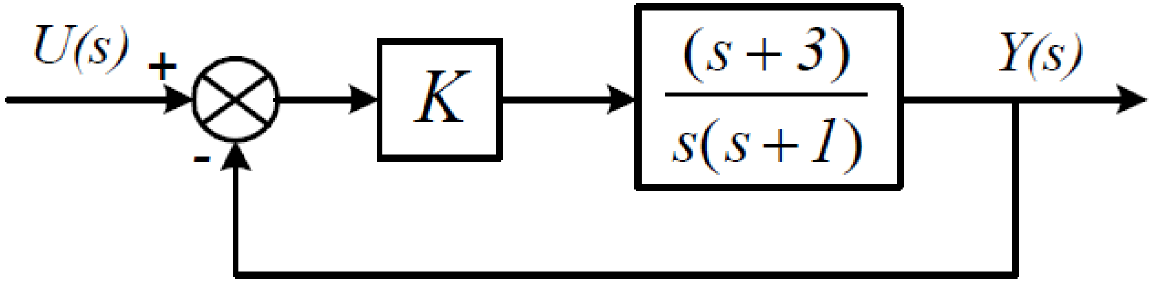
$$= K\left|\frac{-0.6+0.3j+3}{(-0.6+0.3j)(-0.6+0.3j+1)}\right| = 1 \Rightarrow K = 0.1387$$

$$|KP(s_2)| = K\left|\frac{s_2+3}{s_2(s_2+1)}\right| = 1 \Rightarrow K = 6.9547$$

Matlab	$K1=0.1498$
design	$K2=6.8640$

Analytical solution of root locus plot and design

- Find the parameter K which makes the system with a damping ratio of $\zeta=0.866$.



The closed-loop characteristic equation is:

$$s(s+1)+K(s+3)=0$$

$$or \quad s^2+(K+1)s+3K=0$$

$when \quad (K+1)^2-12K \geq 0$ The CLCE has two real roots:

$$s_{1,2}=\frac{-(K+1)\pm\sqrt{(K+1)^2-12K}}{2}$$

Analytical solution of root locus plot and design

when $(K+1)^2 - 12K < 0$ The CLCE has two complex roots:

$$s_{1,2} = \frac{-(K+1) \pm j\sqrt{12K - (K+1)^2}}{2}$$

The solutions to $(K+1)^2 - 12K = 0$ are $K_1=0.101$, $K_2=9.899$

The CL poles are real when $0<K\leq 0.101$ and $K\geq 9.899$

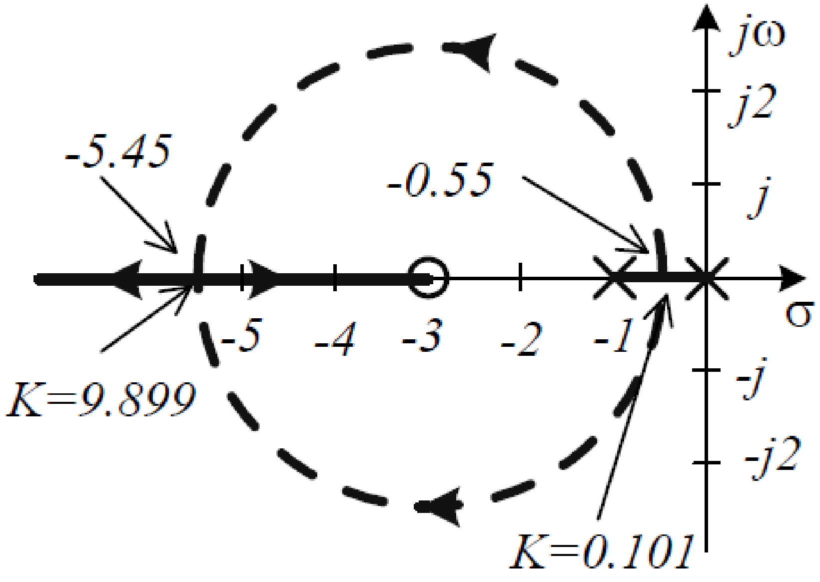
The CL poles are complex when $0.101<K<9.899$

Turning points:

$K = 0.101$, CL poles $s_{1,2}=-0.55$, breakaway point

$K = 9.899$, CL poles $s_{1,2}=-5.45$

break-in point



Analytical solution of root locus plot and design

Now, let's look the exact root locus shape when two poles are complex:

$$s_{1,2} = \frac{-(K+1) \pm j\sqrt{12K-(K+1)^2}}{2} = \sigma \pm j\omega$$

Real Part: $\sigma = \dfrac{-(K+1)}{2}$ Imaginary part : $\omega = \dfrac{\pm\sqrt{12K-(K+1)^2}}{2}$

It can be proved that:

$$(\sigma+3)^2 + \omega^2 = 6$$

Therefore, the root locus is a circle with centre at (*-3*, *j0*) and the radius equal to $\sqrt{6}$, when two poles are complex.

Analytical solution of root locus plot and design

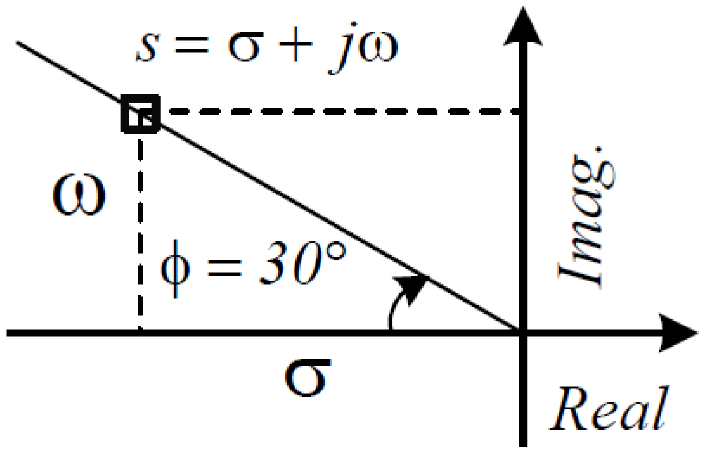
CL poles with the damping ratio *0.866* lie on the line with the angle of $\cos^{-1} 0.866$ ($\approx 30°$) to the negative real axis.

$$\frac{|\omega|}{|\sigma|} = \tan 30° = \frac{\sqrt{3}}{3} \Rightarrow 3\omega^2 = \sigma^2$$

$$\sigma = \frac{-(K+1)}{2} \qquad \omega = \frac{\pm\sqrt{12K-(K+1)^2}}{2}$$

$$3\left[12K-(K+1)^2\right] = (K+1)$$

$$\Rightarrow K = 0.1459 \text{ or } 6.8541$$



when $K=0.1459$, CL poles: $s_1 \approx -0.573 \pm 0.331j$

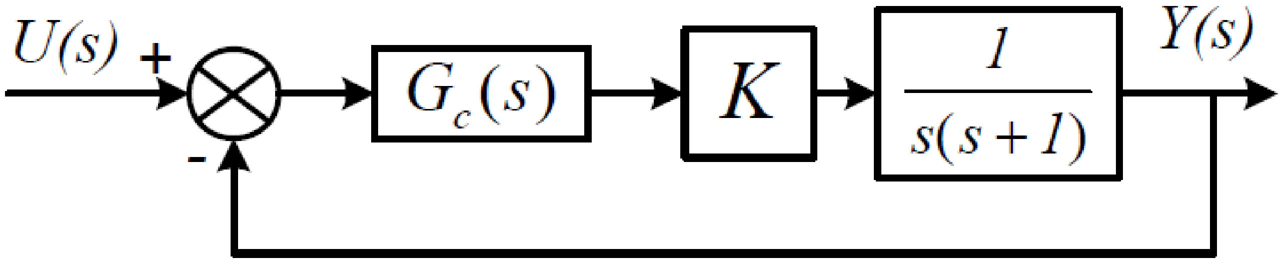
when $K=6.8541$, CL poles: $s_2 \approx -3.927 \pm 2.267j$

Almost identical to the solutions obtained in the last Lecture using the graphic approach.

Root Locus Design Example

For the system shown below, two of the closed-loop poles are to be located at $s=-1.6\pm j4$ using a cascade compensator with transfer function:

$$G_c(s) = \frac{s+2.5}{s+p}$$



Determine: (1) the required value of p, (2) the value of K to locate CL poles required,

Root Locus Design Example

$s = -1.6 \pm j4$ $\quad \zeta\omega_n = 1.6 \quad \omega_n\sqrt{1-\zeta^2} = 4$

$$\tan\phi = \frac{4}{1.6} = 2.5 \Rightarrow \phi = 68.2° \Rightarrow \zeta = \cos\phi = 0.37 \qquad PO = 28.6\%$$

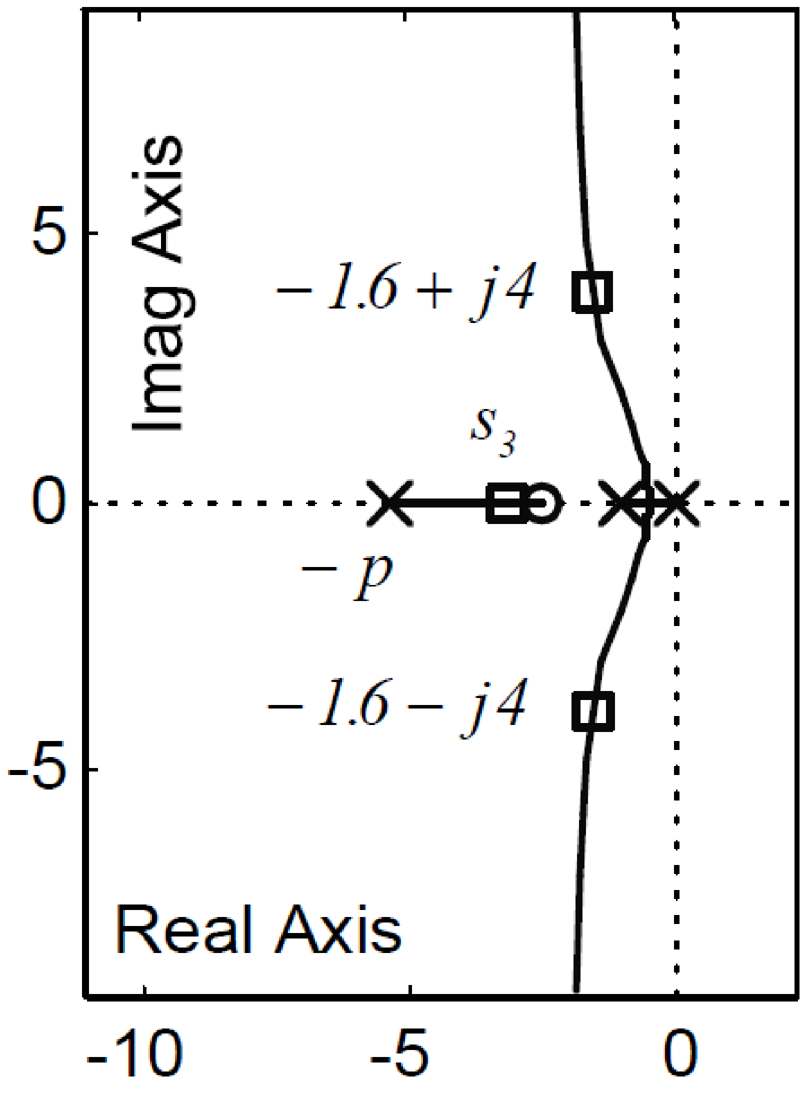
$$t_s \approx \frac{4}{1.6} = 2.5\,\text{sec}$$

The open-loop transfer function is:

$$KP(s) = K\frac{s+2.5}{s(s+1)(s+p)}$$

Root Locus Design Example

Root locus Sketch



Root Locus Design Example

Determine p by using the angle condition:

$$\angle P(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = -180°$$

$$\angle P(s) = \angle(s+2.5) - \angle s - \angle(s+1) - \angle(s+p)$$
$$= -180°$$

for $s = -1.6 + j4$

$$\angle(s+2.5) = \angle(-1.6 + j4 + 2.5) = 77.32°$$
$$\angle s = \angle(-1.6 + j4) = 111.8°$$
$$\angle(s+1) = \angle(-1.6 + j4 + 1) = 98.53°$$

Therefore: $\angle(s+p) = \angle(-1.6 + j4 + p) = 46.99°$

Hence: $\dfrac{4}{p-1.6} = \tan(46.99°) = 1.072 \Rightarrow p = 5.33$

Root Locus Design Example

Determine K by using magnitude condition $|KP(s)| = 1$

$$|KP(s)| = \frac{|s+2.5|}{|s||s+1||s+5.33|}$$

for $s = -1.6 + j4$

$$|KP(s)| = \frac{K|-1.6+j4+2.5|}{|-1.6+j4||-1.6+j4+1||-1.6+j4+5.33|}$$

$$= \frac{K \times 4.1}{4.31 \times 4.04 \times 5.47} = 1$$

Therefore: $K = 23.2$

Step Response Simulation