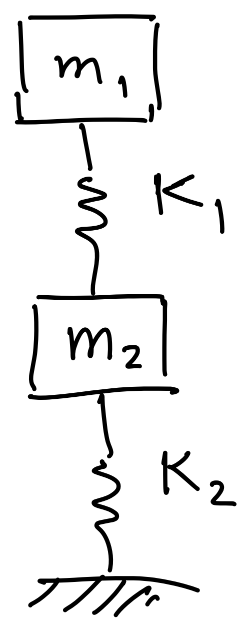
Eigenvalue and Eigenvector problems

There are many applications in engineering and physics.

Many applications in vibration problems.

Example

Find the response of the 2 degree-of-freedom (2 DOF) mass-spring system, when an initial displacement is applied to the mass.

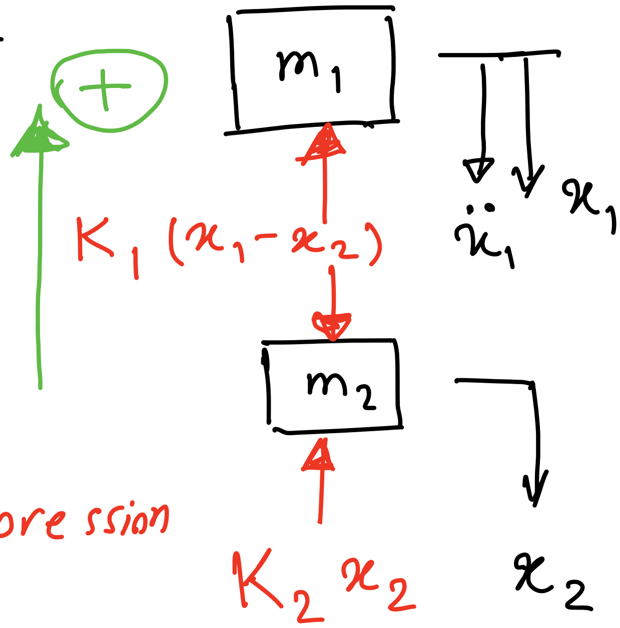


Free-body-diagram (F.B.D):

- Assume the relative motion and direction of motions of m_1 and m_2.

- Let's assume $x_1 > x_2$

(we could assume $x_2 > x_1$, and we get the same answer, in terms of the equations of motion).



$x_1 > x_2 \rightarrow$ Spring in compression

Equations of motions:

using the Newton's second law of motion ($\Sigma F = ma$), and the F.B.D, we have.

For two masses, we need to write two equations (one for each mass).

$$\begin{cases} K_1(x_1 - x_2) = -m_1 \ddot{x}_1 \\ -K_1(x_1 - x_2) + K_2 x_2 = -m_2 \ddot{x}_2 \end{cases}$$

$\ddot{x} = \dfrac{d^2x}{dt}$ acceleration

Let's assume that the solution for x_1

and x_2 are in the form of :

$$x_1 = A_1 e^{i\omega t}$$

$$x_2 = A_2 e^{i\omega t}$$

A_1 is the amplide of the oscillation of mass m_1

A_2 is the amplitude of oscillation of m_2

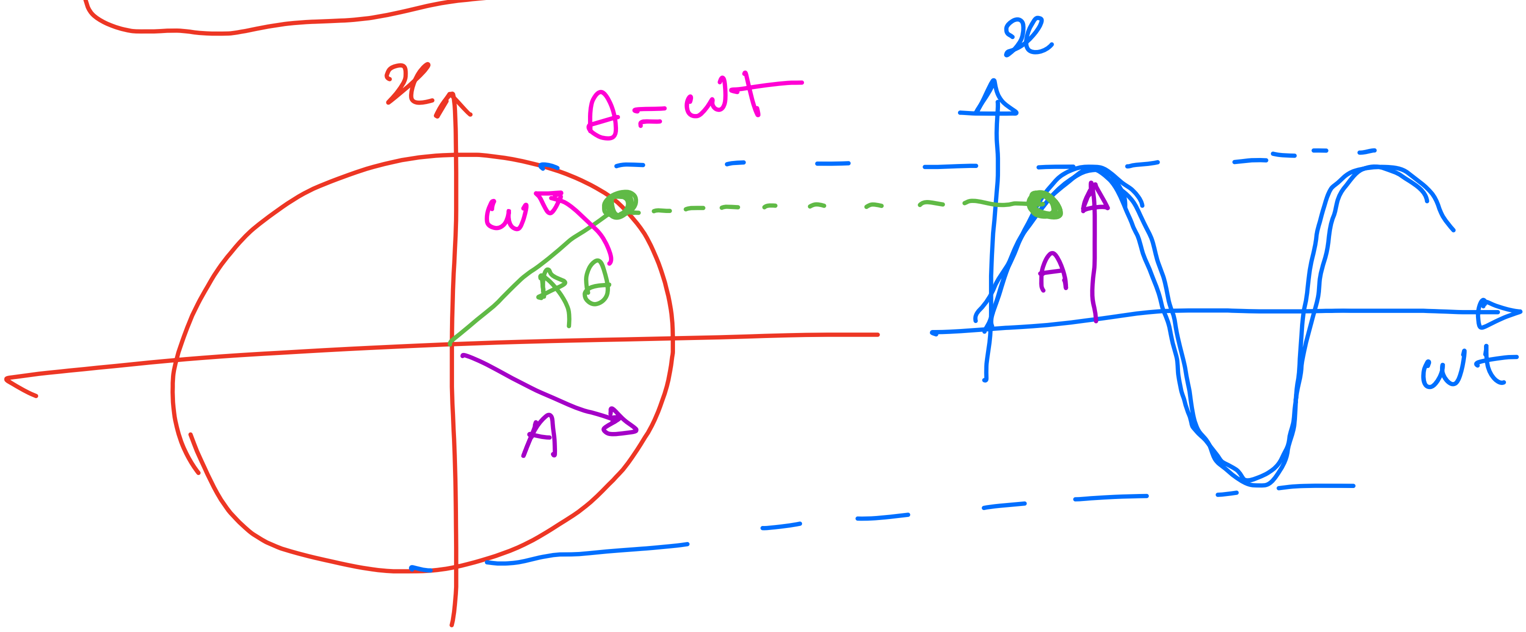
t is time

ω is the angular velocity

To solve for x_1 and x_2 we need to obtain constants A_1 and A_2

$$\text{velocity} = \frac{\text{displacement}}{\text{time}} \qquad \omega = \frac{\theta}{t}$$

$$\theta = \omega t$$



The solutions x_1 and x_2 should satisfy the equations of motion. Therefor, substitute x_1 and x_2 into the equations of motion to obtain A_1 and A_2.

$$x_1 = A_1 e^{i\omega t} \rightarrow \dot{x}_1 = A_1 i\omega e^{i\omega t} \rightarrow \ddot{x}_1 = -A_1 \omega^2 e^{i\omega t}$$

$$x_2 = A_2 e^{i\omega t} \rightarrow \dot{x}_2 = A_2 i\omega e^{i\omega t} \rightarrow \ddot{x}_2 = -A_2 \omega^2 e^{i\omega t}$$

$$\begin{cases} K_1(x_1 - x_2) = -m_1 \ddot{x}_1 \\ -K_1(x_1 - x_2) + K_2 x_2 = -m_2 \ddot{x}_2 \end{cases}$$

substitute

$$\begin{cases} K_1(A_1 - A_2)\cancel{e^{i\omega t}} = -m_1(-A_1\omega^2 \cancel{e^{i\omega t}}) \\ -K_1(A_1 - A_2)\cancel{e^{i\omega t}} + K_2 A_2 \cancel{e^{i\omega t}} = -m_2(-A_2\omega^2 \cancel{e^{i\omega t}}) \end{cases}$$

$$\begin{cases} k_1(A_1 - A_2) = -m_1(-A_1\omega^2) \\ -k_1(A_1 - A_2) + k_2A_2 = -m_2(-A_2\omega^2) \end{cases}$$

Factor A_1 and A_2 :

$$\begin{cases} (\underline{k_1 - m_1\omega^2})A_1 \underline{- k_1}A_2 = 0 \\ \underline{-k_1}A_1 + (\underline{k_1 + k_2 - m_2\omega^2})A_2 = 0 \end{cases}$$

write the equations in matrix form:

$$\underbrace{\begin{bmatrix} k_1 - m_1\omega^2 & -k_1 \\ -k_1 & k_1 + k_2 - m_2\omega^2 \end{bmatrix}}_{M} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If A_1 and A_2 are zero then x_1 and x_2 will be zero. therfore, it is not a good solution.

Therefore, matrix M has to be zero. If a matrix is zero, it means the determinant of the matrix is zero $|M| = 0$.

$$\begin{vmatrix} k_1 - m_1 w^2 & -k_1 \\ -k_1 & k_1 + k_2 - m_2 w^2 \end{vmatrix} = 0$$

$$(k_1 - m_1 w^2)(k_1 + k_2 - m_2 w^2) - (-k_1)(-k_1) = 0$$

If $k_1 = k_2 = 1$ & $m_1 = m_2 = 2$

solve for w.