Differential Transformations of a Frame

The combination of differential translations and rotation in any order.

If we denote the original frame as T and assume that dT is the change in the frame T as a result of a differential transformation,

$$[T + dT] = [\text{Trans}(dx, dy, dz)\, \text{Rot}(\vec{q}, d\theta)][T]$$

$$[dT] = [\text{Trans}(dx, dy, dz)\, \text{Rot}(\vec{q}, d\theta) - I][T]$$

where I is a unit matrix.

$$[dT] = [\Delta][T]$$

$$[\Delta] = [\mathrm{Trans}(dx, dy, dz)\,\mathrm{Rot}(\vec{q}, d\theta) - I]$$

$[\Delta]$ (or simply Δ) is called differential operator.

$$\Delta = \mathrm{Trans}(dx, dy, dz)\,\mathrm{Rot}(\vec{q}, d\theta) - I$$

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

write the differential operator matrix for the following differential transformations:

$dx = 0.05$, $dy = 0.03$, $dz = 0.01$ units

and $\delta x = 0.02$, $\delta y = 0.04$, $\delta z = 0.06$ radians.

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -0.06 & 0.04 & 0.05 \\ 0.06 & 0 & -0.02 & 0.03 \\ -0.04 & 0.02 & 0 & 0.01 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Find the effect of a differential rotation of 0.1 rad about the y-axis followed by a differential translation of [0.1, 0, 0.2] on the given frame B.

$$B = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$dx = 0.1 \quad dy = 0 \quad dz = 0.2$

$\delta x = 0 \quad \delta y = 0.1 \quad \delta z = 0$

$$[dB] = [\Delta][B]$$

$$[dB] = \begin{bmatrix} 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times B$$

$$[dB] = \begin{bmatrix} 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpretation of the differential change

$$dT = \begin{bmatrix} dn_x & do_x & da_x & dp_x \\ dn_y & do_y & da_y & dp_y \\ dn_z & do_z & da_z & dp_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The new location and orientation of a frame after the differential motions:

$$T_{new} = dT + T_{old}$$

Example (3.6 Book)

Find the location and the orientation of frame B of the previous example, after the move

$$B_{new} = B_{original} + dB$$

$$= \begin{bmatrix} 0 & 0.1 & 1 & 10.4 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -0.1 & 2.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential changes between frames

The differential operator Δ in Equation (5.16) represents a differential operator relative to the fixed reference frame and is technically ${}^{U}\Delta$.

However, it is possible to define another differential operator, this time relative to the current frame itself.

Since the differential operator relative to the frame ($^T\Delta$) is relative to a current frame, to find the changes we post-multiply the frame by $^T\Delta$.

The result will be the same, since both operations represent the same changes in the frame. Then:

$$[dT] = [\Delta][T] = [T][^T\Delta]$$

$$[\Delta][T] = [T][^T\Delta]$$

$$[T^{-1}][\Delta][T] = \underbrace{[T^{-1}][T]}_{I}[^T\Delta]$$

$$[{}^{T}\Delta] = [T^{-1}][\Delta][T]$$

$$T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\vec{p}\cdot\vec{n} \\ o_x & o_y & o_z & -\vec{p}\cdot\vec{o} \\ a_x & a_y & a_z & -\vec{p}\cdot\vec{a} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^{T}\Delta = [T^{-1}][\Delta][T]$$

$${}^{T}\Delta = \begin{bmatrix} 0 & -{}^{T}\delta z & {}^{T}\delta y & {}^{T}dx \\ {}^{T}\delta z & 0 & -{}^{T}\delta x & {}^{T}dy \\ -{}^{T}\delta y & {}^{T}\delta x & 0 & {}^{T}dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$^T\Delta$ is made to look exaclty like the Δ matrix. but all elements are relative to the current frame, where these elements are found from the above multiplication of matrices, and are as fllows:

$$^T\delta x = \vec{\delta} \cdot \vec{n}$$

$$^T\delta y = \vec{\delta} \cdot \vec{o}$$

$$^T\delta z = \vec{\delta} \cdot \vec{a}$$

$$^Tdx = \vec{n} \cdot [\vec{\delta} \times \vec{p} + \vec{d}]$$

$$^Tdy = \vec{o} \cdot [\vec{\delta} \times \vec{p} + \vec{d}]$$

$$^Tdz = \vec{a} \cdot [\vec{\delta} \times \vec{p} + \vec{d}]$$

Example

Find ${}^{B}\Delta$ for the previous example.

$$B = \begin{pmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{n} = (n_x, n_y, n_z) = n_x \vec{i} + n_y \vec{j} + n_z \vec{k}$$

$$\vec{n} = [0, 1, 0]$$

$$\vec{o} = [0, 0, 1]$$

$$\vec{a} = [1, 0, 0]$$

$$\vec{P} = [10, 5, 3]$$

$$\vec{\delta} = [0, 0.1, 0]$$

$$\vec{d} = [0.1, 0, 0.2]$$

$$\vec{\delta} \times \vec{P} = \begin{vmatrix} i & j & k \\ 0 & 0.1 & 0 \\ 10 & 5 & 3 \end{vmatrix} = [0.3, 0, -1]$$

$$\vec{\delta} \times \vec{P} + \vec{d} = [0.3, 0, -1] + [0.1, 0, 0.2]$$
$$= [0.4, 0, -0.8]$$

$${}^{T}\delta x = \vec{\delta} \cdot \vec{n} = 0(0) + 0.1(1) + 0(0) = 0.1$$

$${}^{T}\delta y = \vec{\delta} \cdot \vec{o} = 0(0) + 0.1(0) + 0(1) = 0$$

$${}^{T}\delta z = \vec{\delta} \cdot \vec{a} = 0(1) + 0.1(0) + 0(0) = 0$$

$${}^{T}dx = \vec{n} \cdot [\vec{\delta} \times \vec{P} + \vec{d}] = 0(0.4) + 1(0) + 0(-0.8)$$

$${}^{T}dy = \vec{o} \cdot [\vec{\delta} \times \vec{P} + \vec{d}] = 0(0.4) + 0(0) + 1(-0.8)$$
$$= -0.8$$

$${}^{T}dz = \vec{a} \cdot [\vec{\delta} \times \vec{P} + \vec{d}] = 1(0.4) + (0)(0) + 0(-0.8)$$
$$= 0.4$$

$$ {}^{B}\Delta = \begin{bmatrix} 0 & -{}^{T}\delta z & {}^{T}\delta y & {}^{T}d x \\ {}^{T}\delta z & 0 & -{}^{T}\delta x & {}^{T}d y \\ -{}^{T}\delta y & {}^{T}\delta x & 0 & {}^{T}d z \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

$$ {}^{B}\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

Example

Calculate ${}^{B}\Delta$ of the previous example directly from the differential operator

(Equation 3.20)

$$ [{}^{B}\Delta] = [B^{-1}][\Delta][B] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

Next week

Calculation of Jacabian