Instrumentation and Controls
ETM 3301

Lecture 18

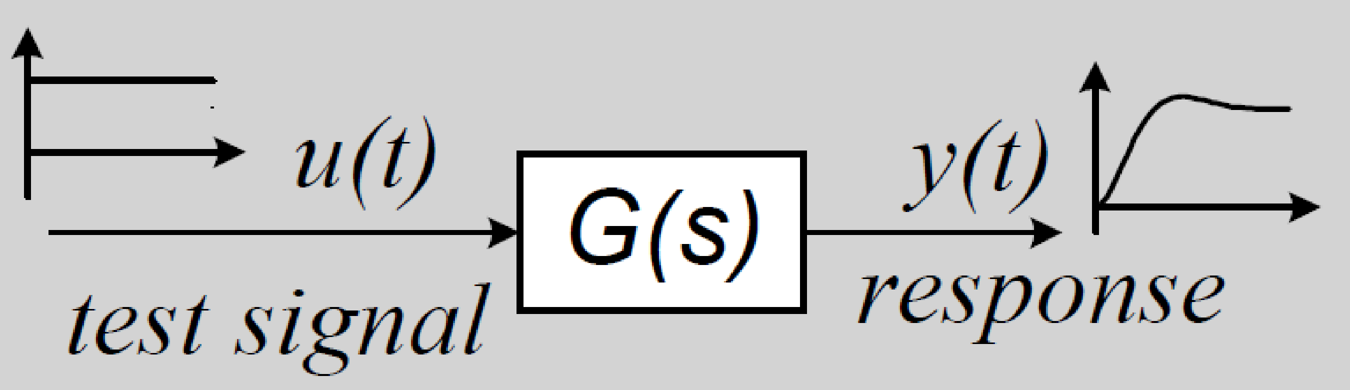
Instructor
Dr. Farbod Khoshnoud

Chapter 7: Root Locus Method

- Importance/Relevance of pole (root) locations.
- What is the root locus of a feedback control system.
- Conditions a root locus should satisfies.
- Important root locus construction rules.
- Control system design using root locus.

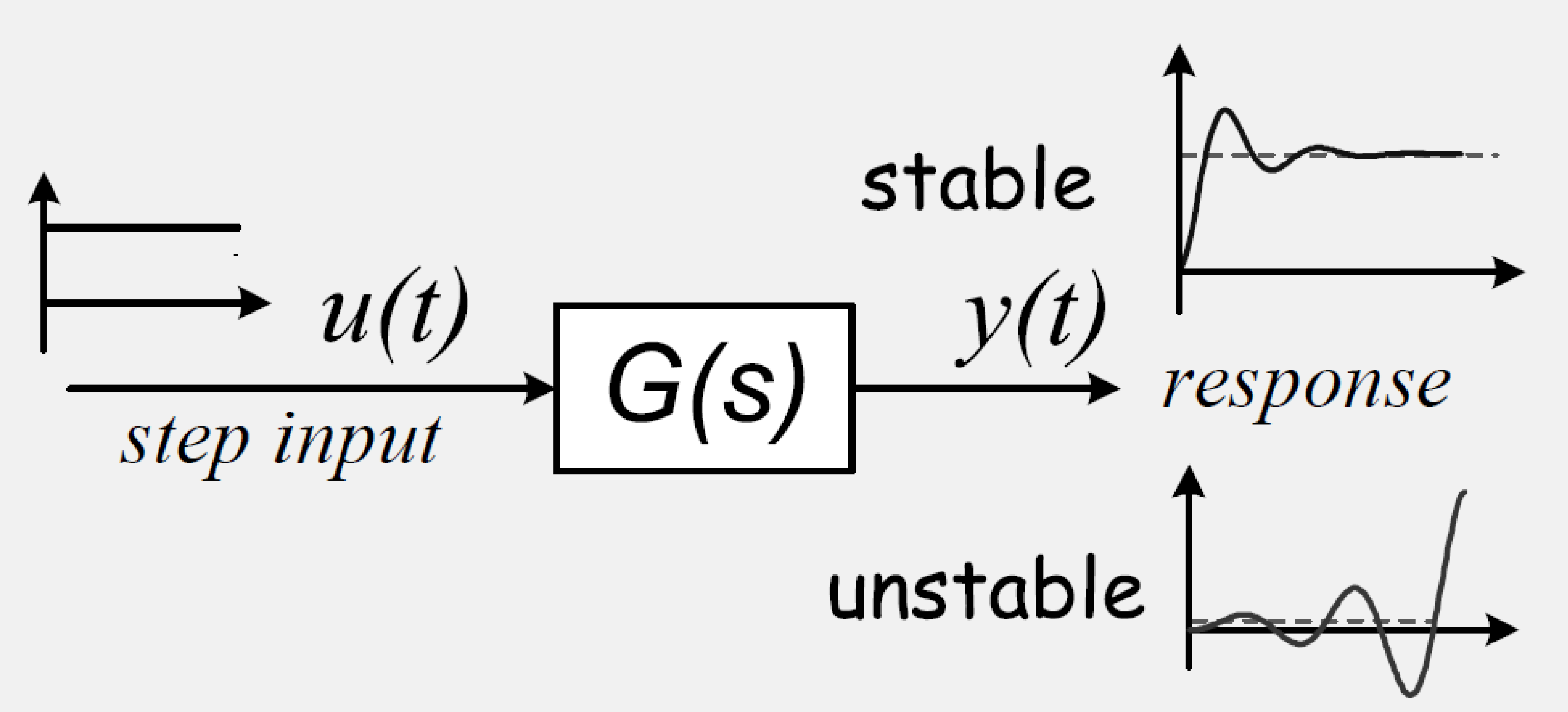
Control System & Transfer Function

- One of main purposes of control is to make output follow input.
- Transfer functions and test signals are used to study control system properties.



$$\frac{Y(s)}{U(s)} = G(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Stable and Unstable System Step Responses

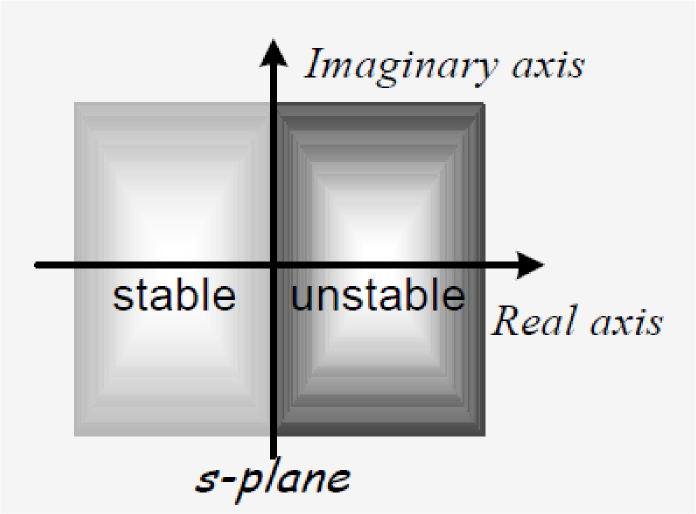


- Stable system:
  - the response gradually settles down to a required value.

Stability and Pole Location

- A system is stable *if and only if* all system poles lie in the left hand side of *s*-plane, i.e.

$$\mathrm{Re}(poles) = \mathrm{Re}(-p_i) < 0 \quad i = 1, 2, \cdots n$$



- To verify the system stability, we need to find all system poles.

What are Poles? A Revision

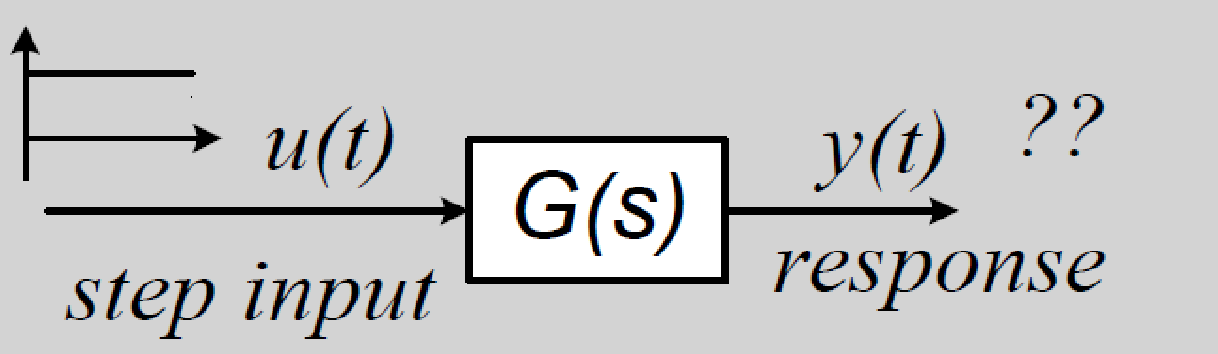
$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

D(s)	the **characteristic polynomial** (CP) of *G(s)*
D(s)=0	the **characteristic equation** (CE) of *G(s)*
poles	the solutions (roots) of the equation *D(s)=0*
zeros	the solution (roots) of the equation *N(s)=0*

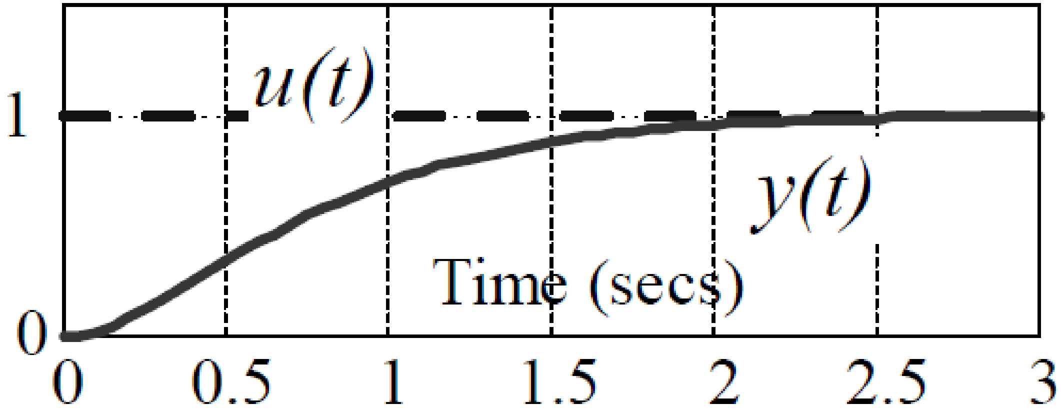
$$G(s) = \frac{3}{s^2 + 4s + 3} = \frac{3}{(s+1)(s+3)}$$

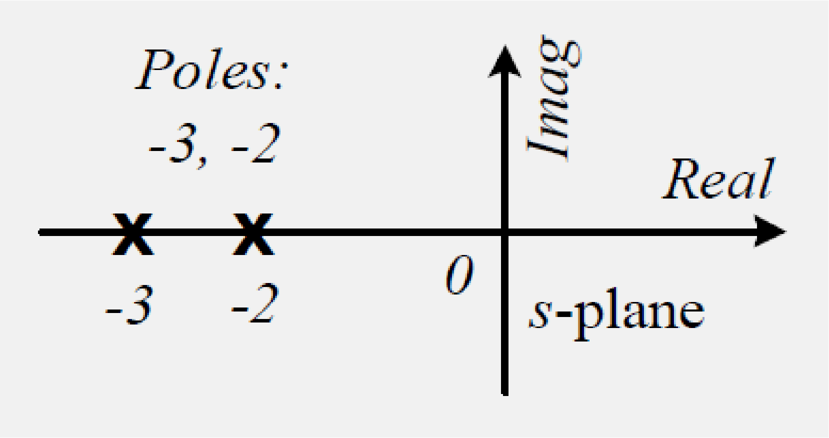
$$CE: \quad (s+1)(s+3) = 0, \quad \text{poles}: -1, -3$$

Pole position and step response (1)



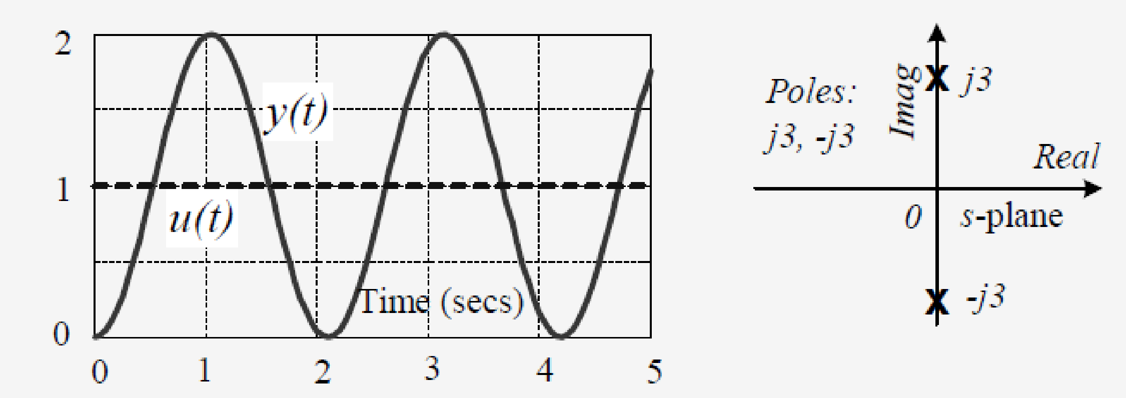
$$G(s) = \frac{6}{s^2 + 5s + 6} = \frac{6}{(s+2)(s+3)}$$



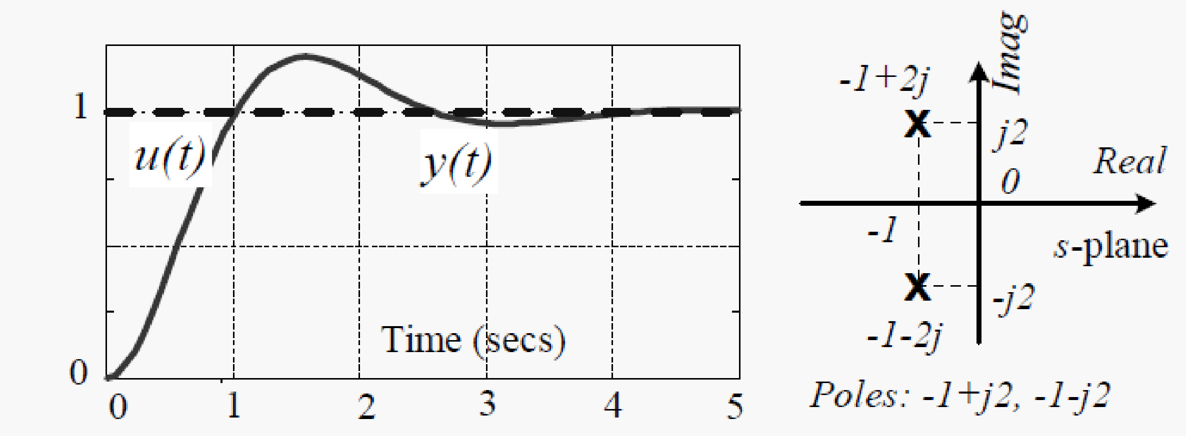


Pole position and step response (2)

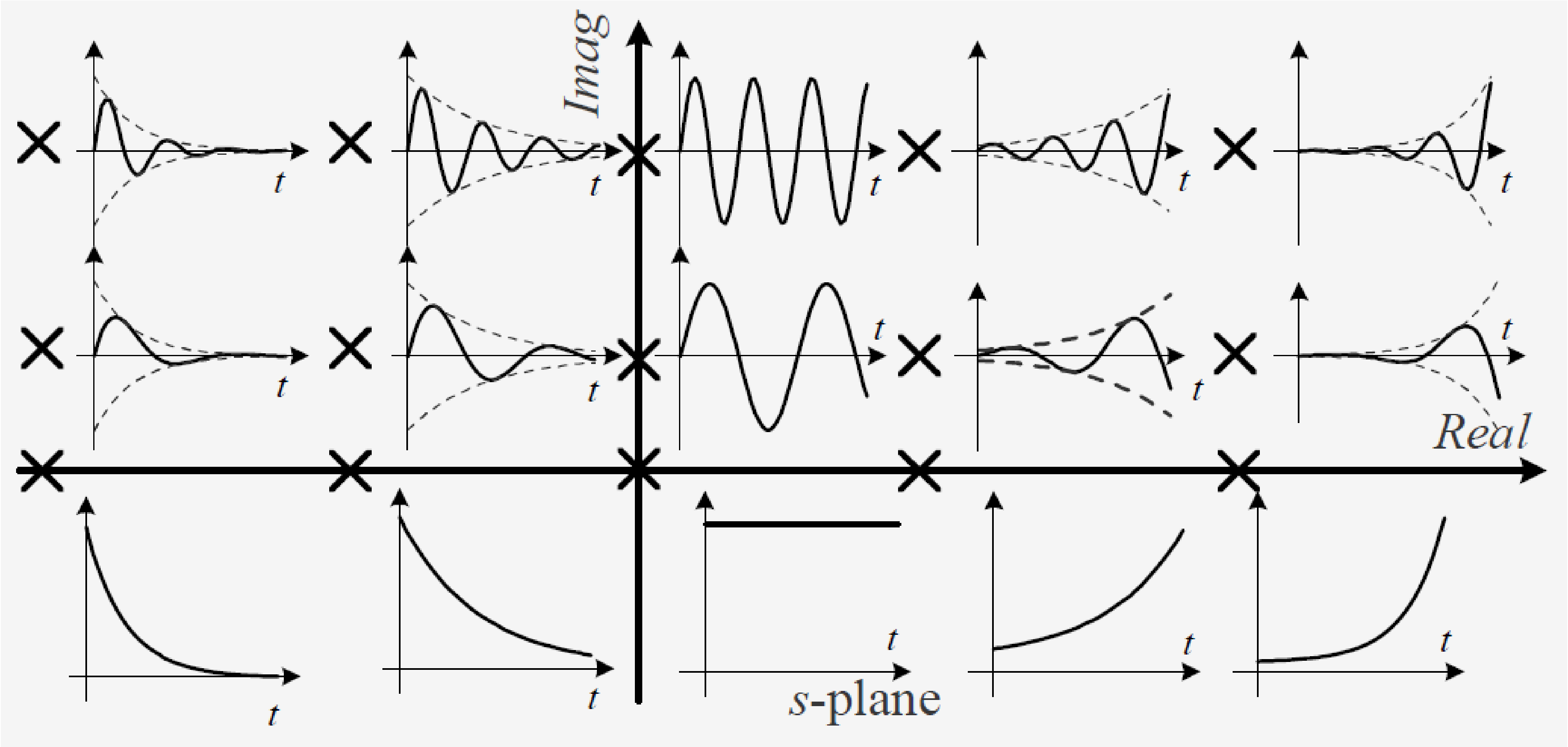
$$G(s) = \frac{9}{s^2+9} = \frac{9}{(s-j3)(s+j3)}$$



$$G(s) = \frac{5}{s^2+2s+5} = \frac{5}{(s+1-j2)(s+1+j2)}$$



Pole locations and transient responses



- System response characteristics determined by transfer function pole locations.

Importance of Pole Locations: Summary

- Stability condition: $\text{Re}\{\text{all poles}\}<0$.
- Transient response: directly related to the locations of the closed-loop system poles in the *s*-plane.
- Pole locations: very important for control systems analysis and design.

Pole locations and transient response performance indices ,1

- 2nd order system $$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

Poles: $$s_{1,2} = -\zeta\omega_n \pm j\omega_d \quad when \quad 0 \le \zeta \le 1$$
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

Transient (dynamic) performance indices:

$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \qquad t_s = \frac{4}{\zeta\omega_n}$$

PO depends on ζ

t_s depends on $\zeta\omega_n$

Pole locations and transient response performance indices ,2

Poles:

$-\zeta\omega_n \pm j\omega_d$

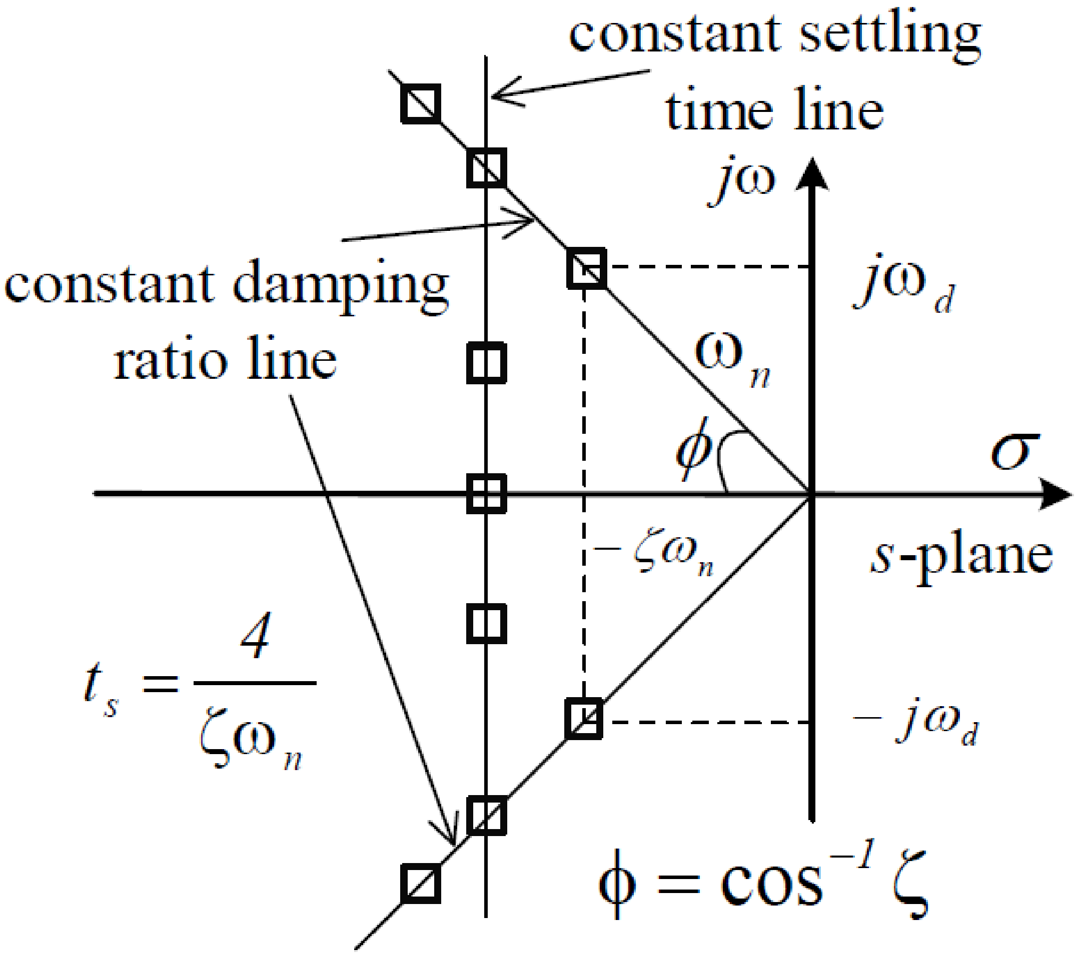
$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

Real part: $-\zeta\omega_n$

Settling time depends on real part

$\phi = \cos^{-1}\zeta$

Damping ratio depends on angle ϕ

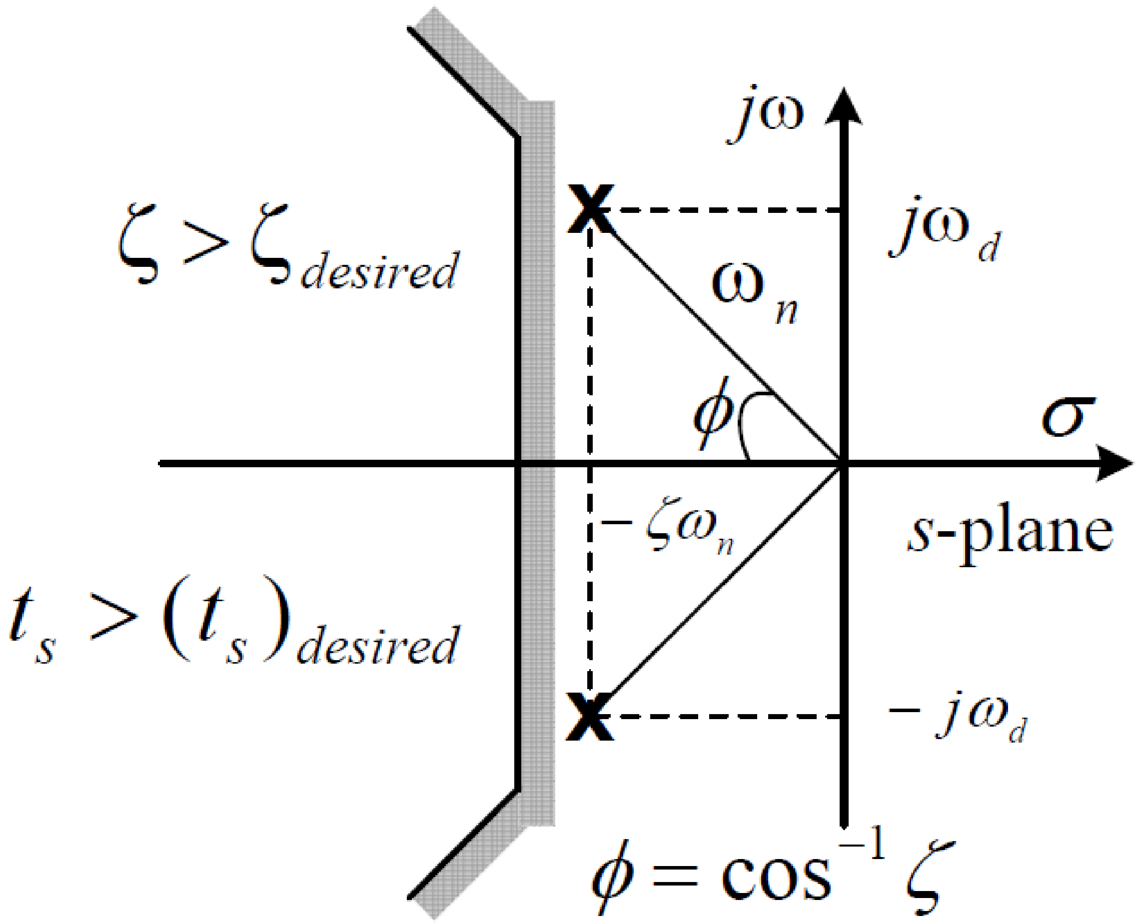


Pole locations and performance indices

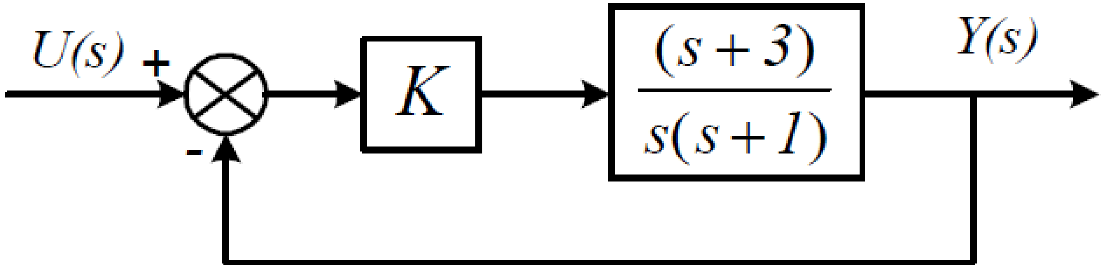
- Percentage overshoot:
  - Depends on the angle ϕ which relates to the relative sizes of real and imaginary parts.
  - The smaller the angle ϕ, the larger the damping ratio ζ and hence the smaller the PO.
- Settling time:
  - Depends on the real part of poles.
  - The bigger the real part, the smaller the t_s.

$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \qquad t_s = \frac{4}{\zeta\omega_n}$$

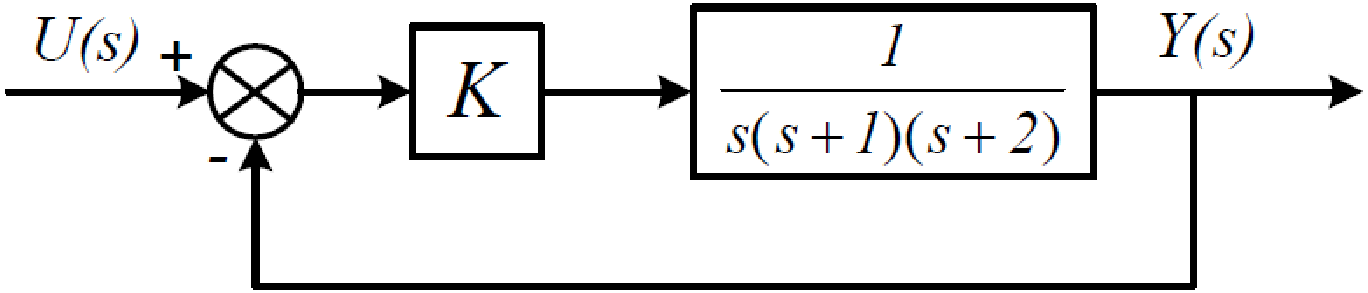
Control System Design by Place Poles on Complex Plane



Control System Design by Parameter Adjustment
Higher Order & No-standard 2nd order Systems



$$\frac{s+3}{s^2+(1+K)s+3K}$$



$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

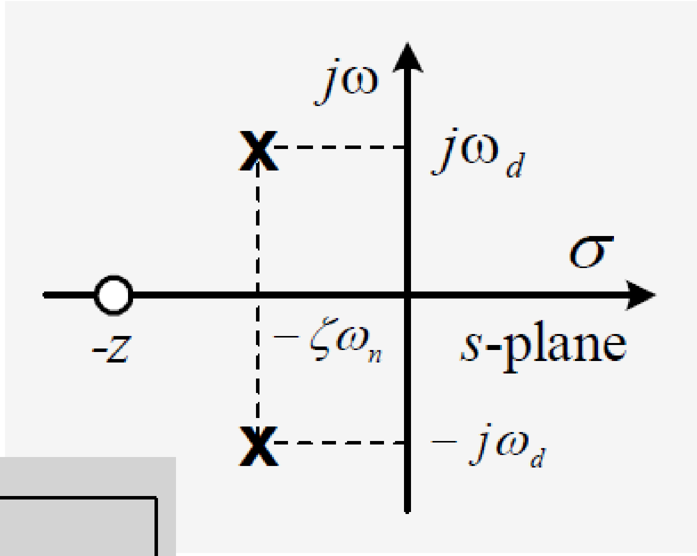
$$\frac{K}{s^3+3s^2+2s+K}$$

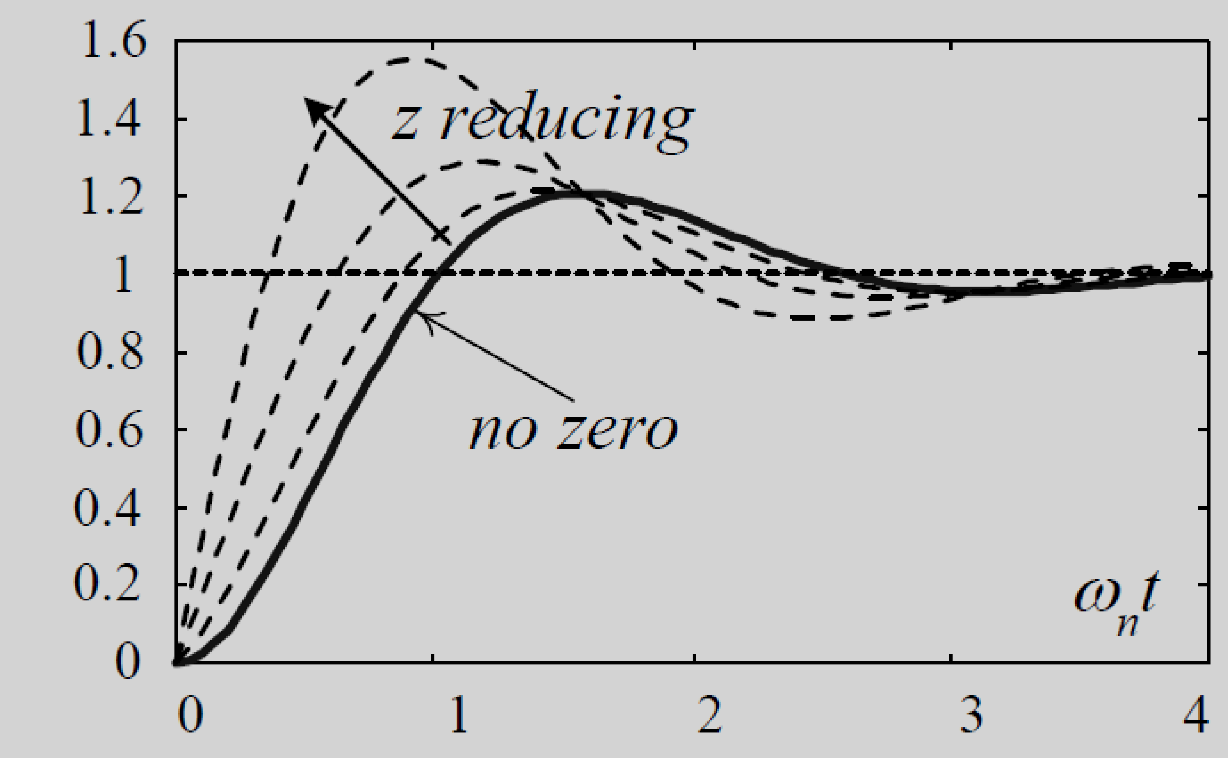
2nd order system percentage Overshoot and settling time formula *cannot* be used in design!

Second Order + Additional Zero

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{s+z}{z}\right)$$

$$0 < \zeta < 1, \quad z > 0$$



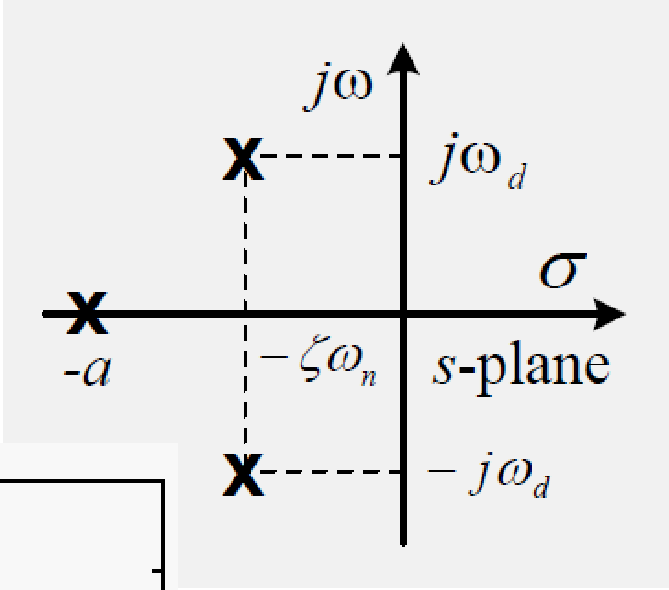


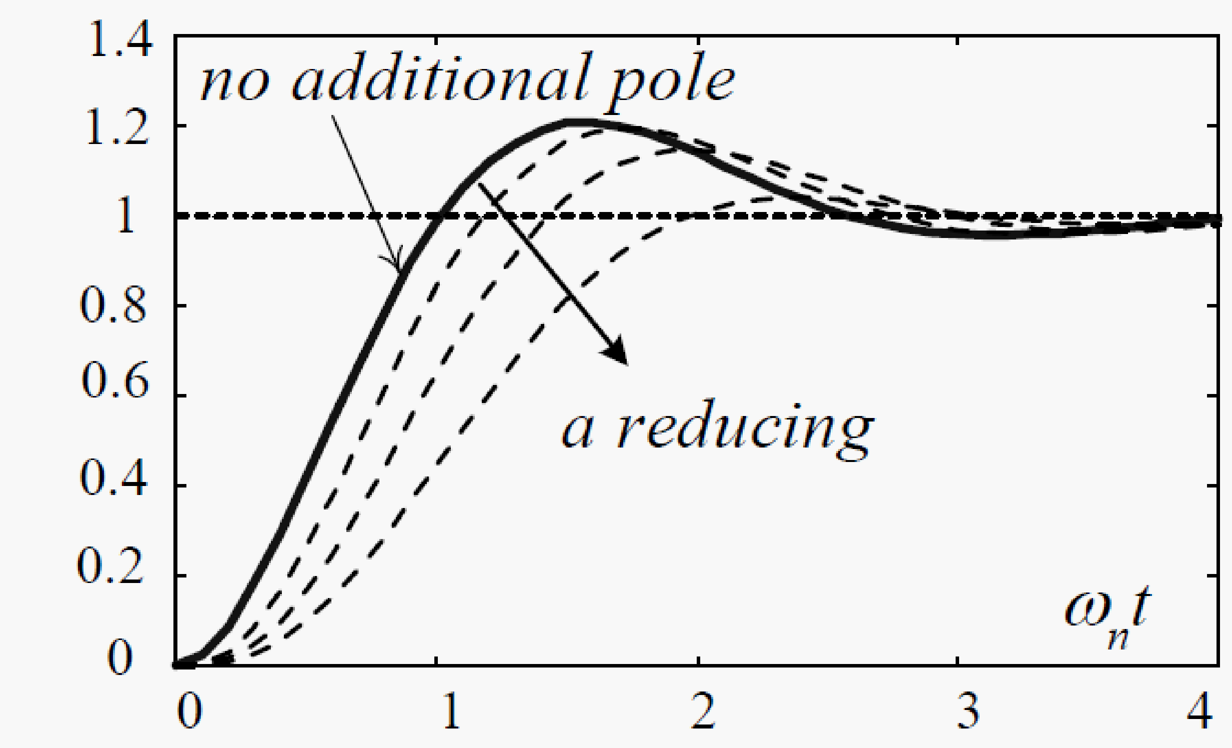
Bigger z, smaller difference in responses.

Second Order + Additional Pole

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{a}{s+a}\right)$$

$$0 < \zeta < 1, \quad a > 0$$





Bigger a, smaller difference in responses.

A Pair of Dominant Poles

- For systems with the order higher than one, the dynamic characteristics can be approximated by a pair of *dominant poles*.
- A pair of *dominant poles* are dominate if real parts of all zeros and other poles are significantly larger than real parts of dominant poles.
  - At least 3 times bigger.

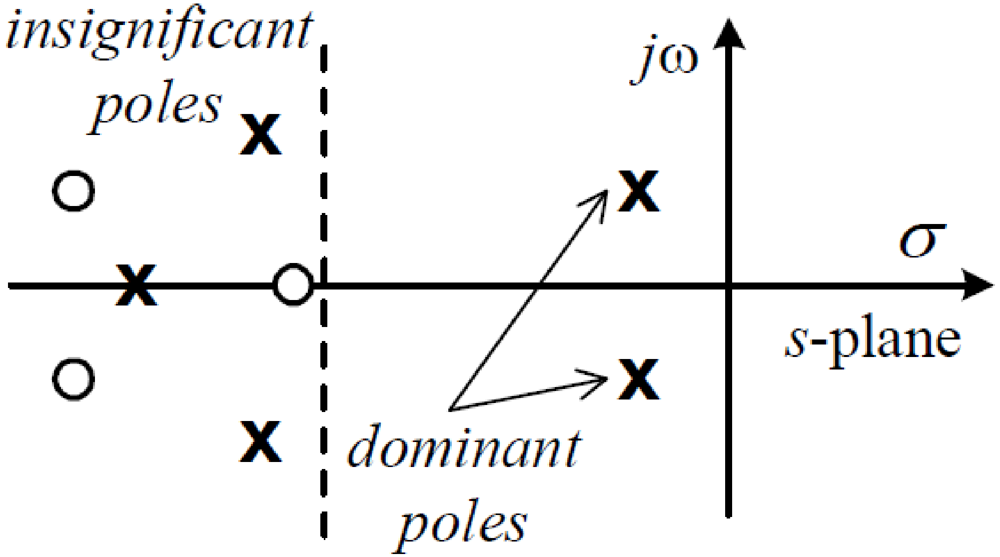
Poles: $-1 \pm 2j, -4, -8$ Dominant Poles: $-1 \pm 2j$

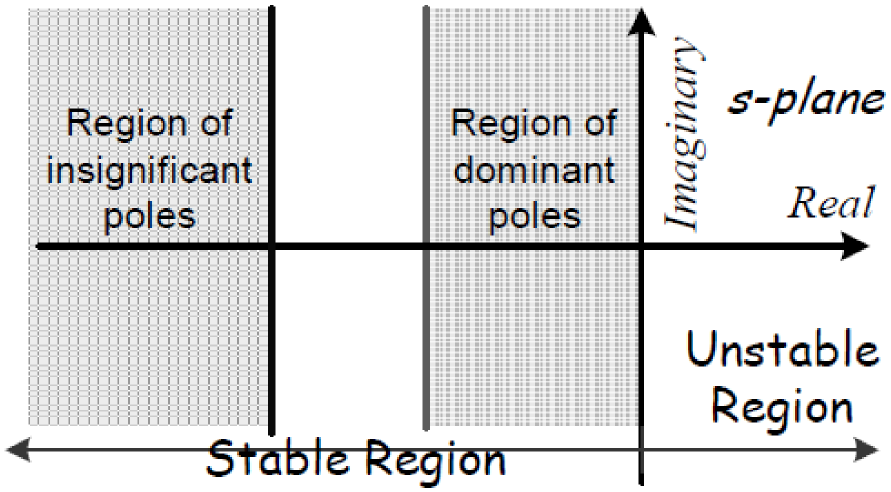
- When we have a pair of *dominant poles*, we can applied concepts of standard 2nd order systems for designing higher order systems.

Second order system approximation

- A system with more than two poles or with zeros can be approximated as a system that has just two complex ***dominant poles***.
  - If the real parts of zeros and the real parts of other poles are significantly larger than the real part of these two dominant poles.

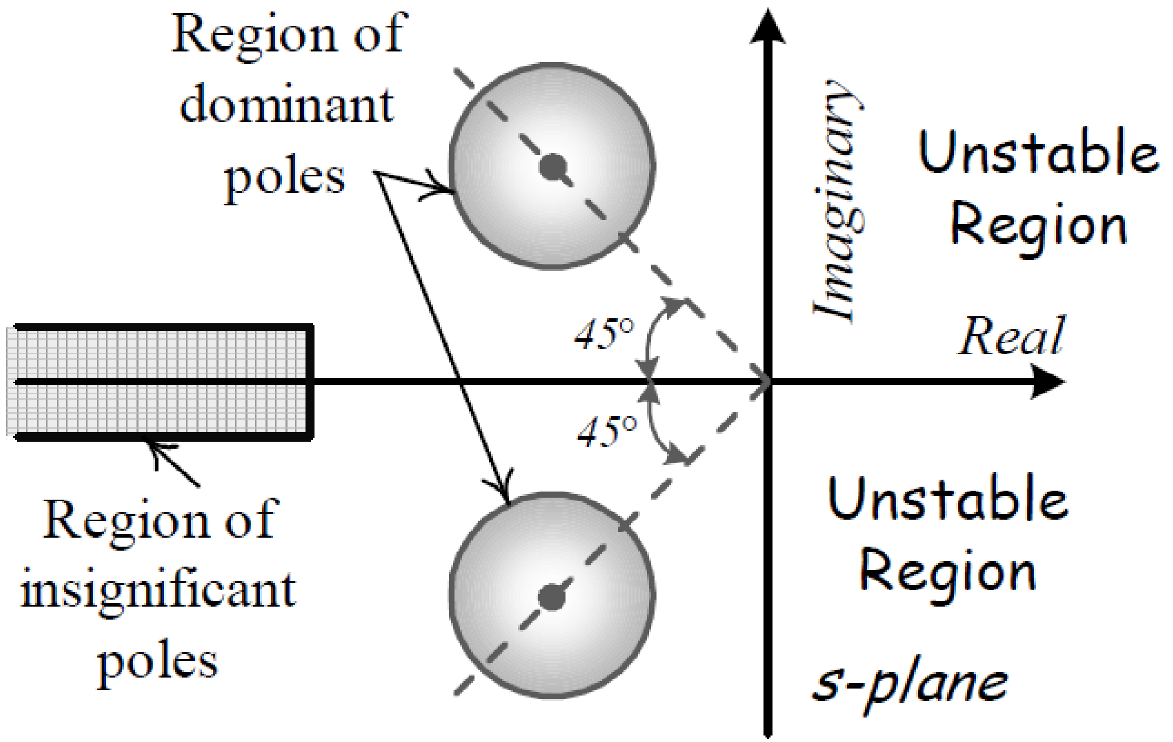
- How large is significantly large?
  - At least 3 times.





Dominant Poles

Optimal design: damping ratio of dominant poles is around *0.707* *(PO≈4%)*



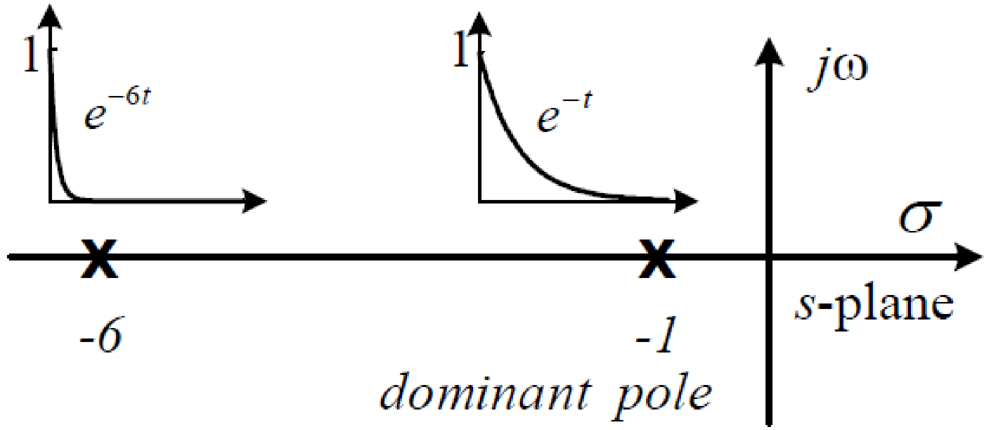
Dominant Pole Example (All poles real)

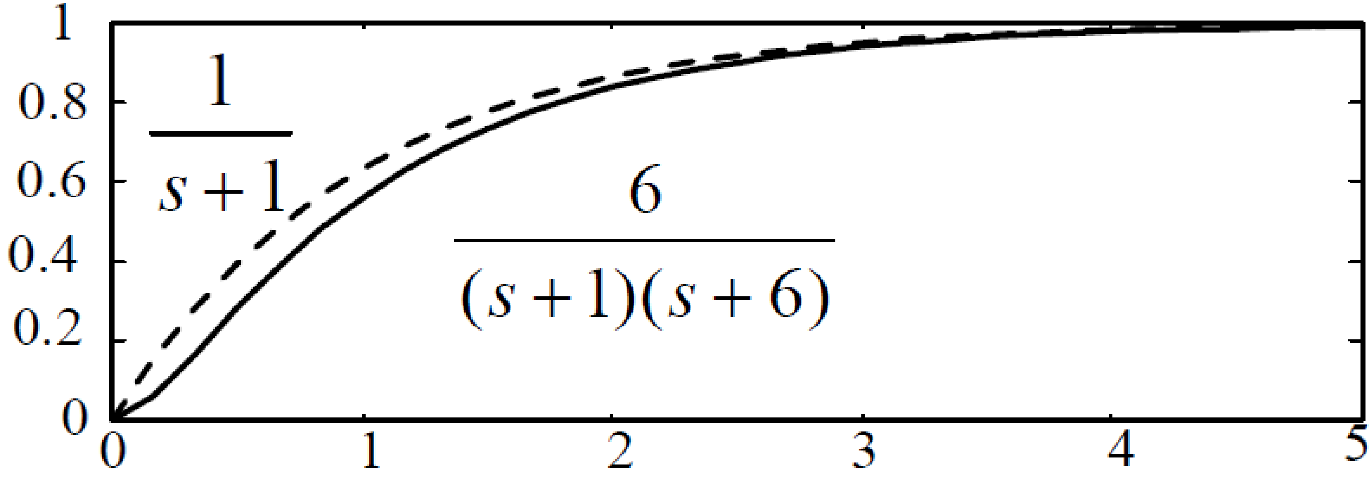
$$G(s) = \frac{6}{(s+1)(s+6)}$$

Unit step response

$$y(t) = 1 - \frac{6}{5}e^{-t} + \frac{1}{5}e^{-6t}$$

The transient response due to pole at $s=-1$ will persist much longer than the transient response due to pole at $s=-6$.

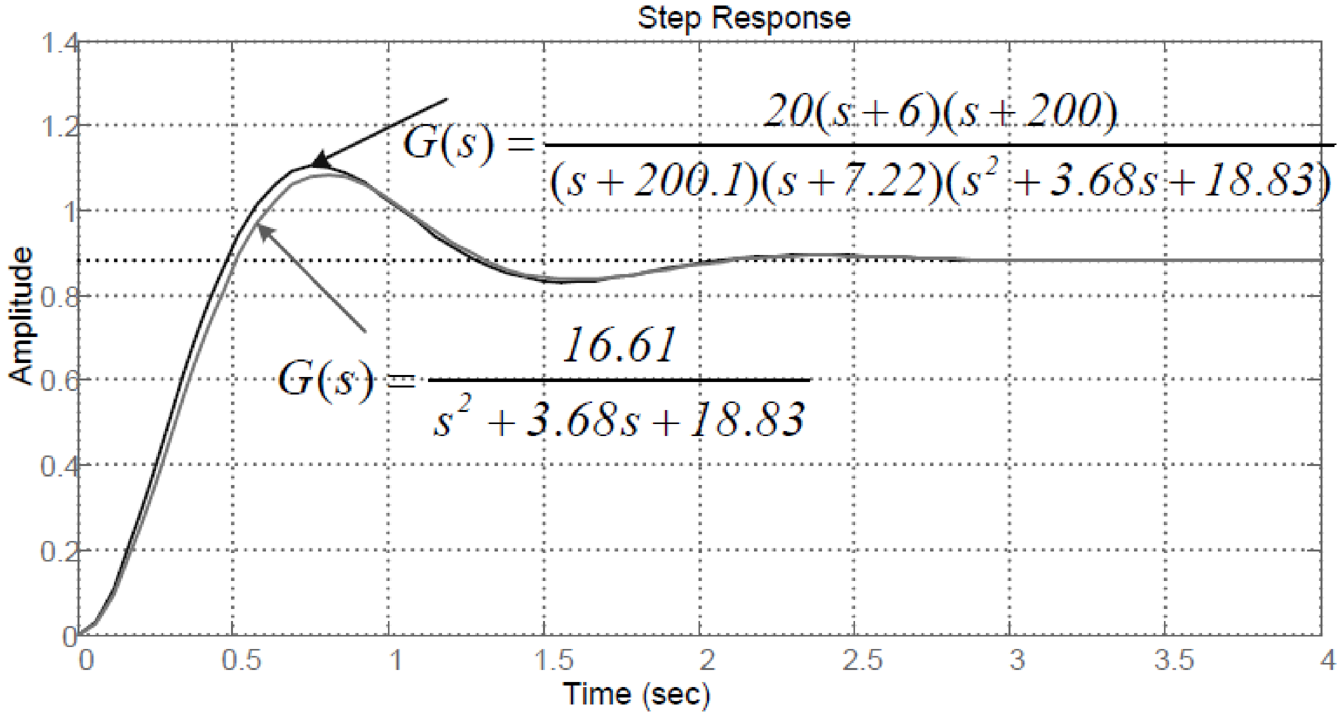




Dominant Pole Example: dominate complex poles)

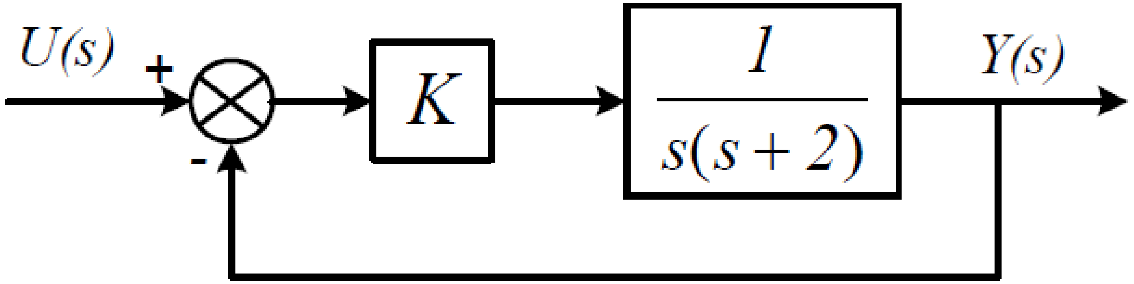
$$G(s) = \frac{20(s+6)(s+200)}{(s+200.1)(s+7.22)(s^2+3.68s+18.83)}$$

Zeros: *-200*, *-6*. Poles: *-200.1*, *-7.22*, *-1.84±3.93j*. The pair of complex poles *-1.84±3.93j* dominate because real parts of zeros and other poles are at least three times bigger.



Pole Location Adjustment Example (1-1)

- Suitable pole positions can be obtained by adjusting one or more system parameters.



- It is worthwhile to find out the paths of poles on s-plane when the parameter K varies from 0 to $+\infty$.

Poles: roots of characteristic equation.

Pole Location Adjustment Example (1-2)

Closed-loop transfer function (CLTF):

$$T(s) = \frac{K\dfrac{1}{s(s+2)}}{1 + K\dfrac{1}{s(s+2)} \times 1} = \frac{K}{s^2 + 2s + K}$$

Closed-loop characteristic equation (CLCE):

$$s^2 + 2s + K = 0$$

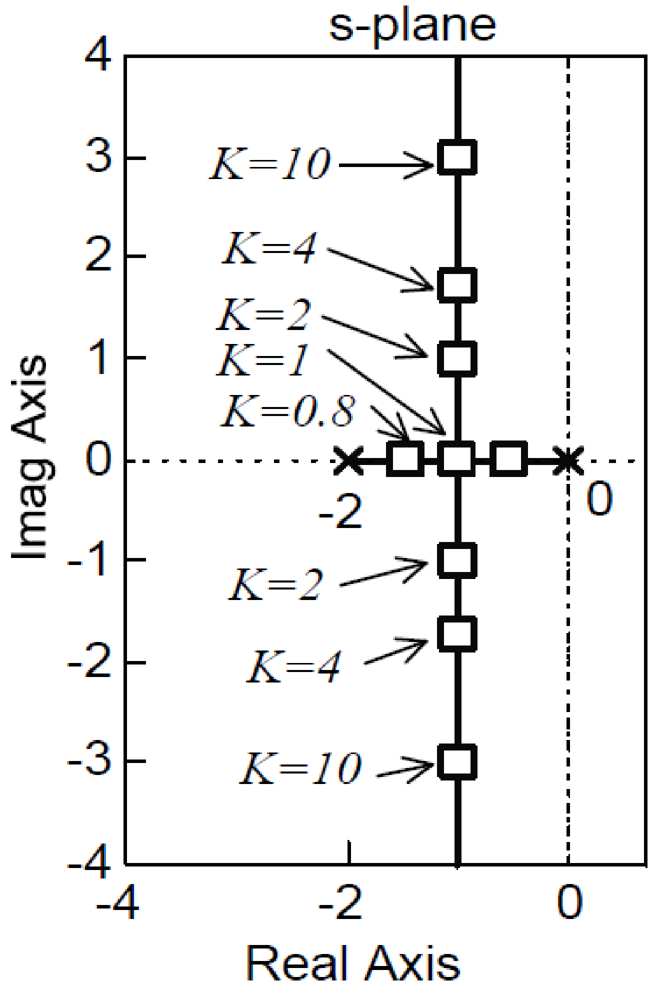
$$s_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4K}}{2} = \frac{-2 \pm \sqrt{4(1-K)}}{2} = \frac{-2 \pm 2\sqrt{1-K}}{2} = -1 \pm \sqrt{1-K}$$

$$\begin{aligned} Poles: \quad & s_{1,2} = -1 \pm \sqrt{1-K} \qquad 0 < K \le 1 \\ & s_{1,2} = -1 \pm j\sqrt{K-1} \qquad K > 1 \end{aligned}$$

Pole Location Adjustment Example (1-3)

- Closed-loop poles (Roots of CE) as the parameter K changes.

K	poles (roots)
0	$0, -2$
0.8	$-1.45, -0.55$
1	$-1, -1$
2	$-1 \pm 1j$
4	$-1 \pm 1.73j$
10	$-1 \pm 3j$
∞	$-1 \pm \infty j$



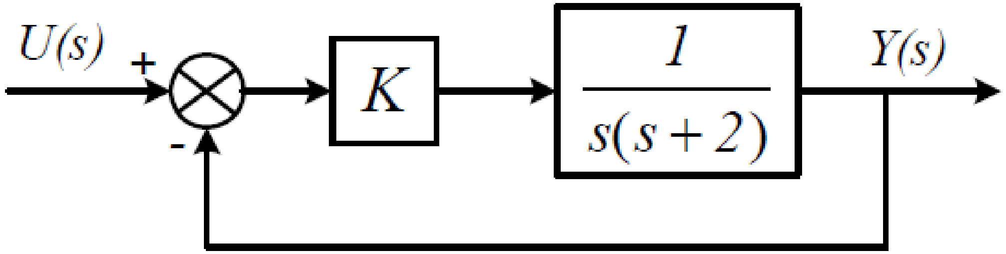
Trace of poles as K varies

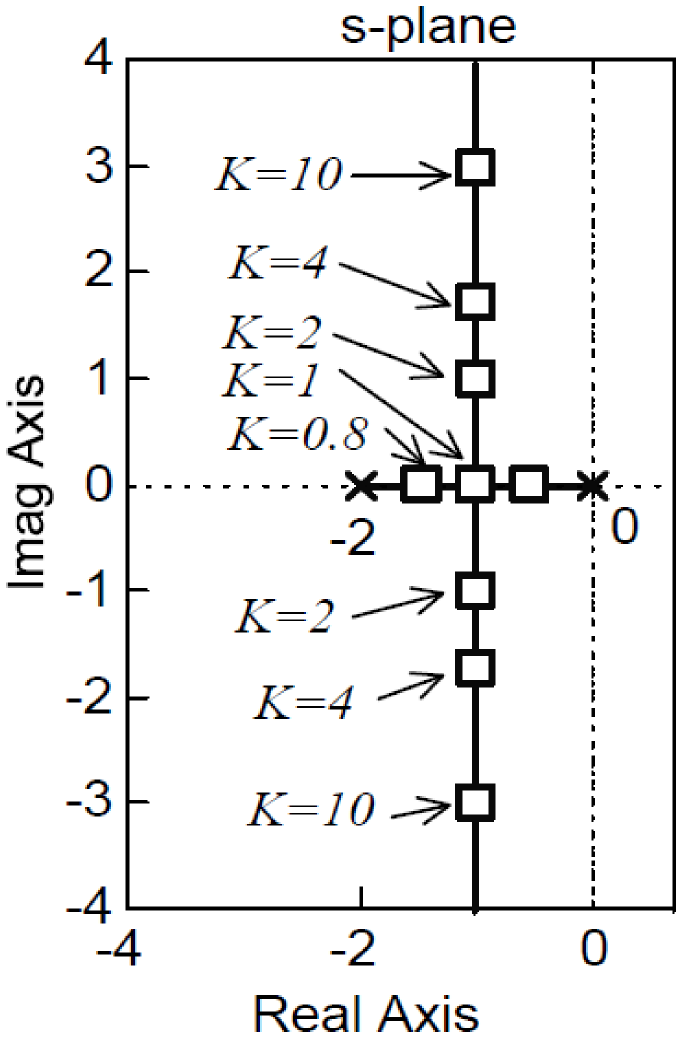
$$\text{Poles}: \quad s_{1,2} = -1 \pm \sqrt{1-K} \qquad 0 < K \le 1$$

$$s_{1,2} = -1 \pm j\sqrt{K-1} \qquad K > 1$$

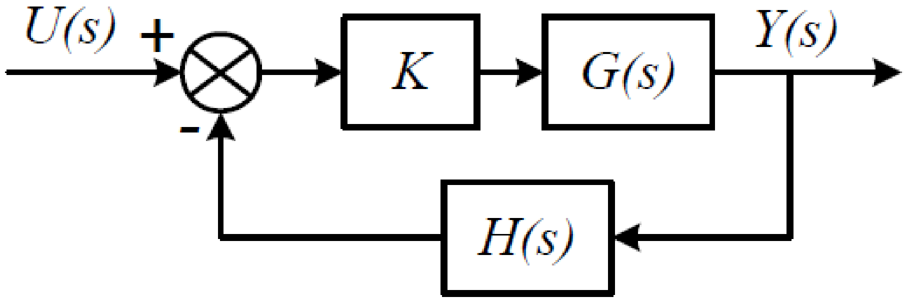
Root Locus Definition

The ***root locus*** is a plot of the locations of the roots of a characteristic equation on s-plane as a system parameter varies from *0* to $+\infty$.





Root Locus Terms



Closed-loop transfer function (CLTF):

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

Closed-loop characteristic equation (CLCE):

$$1 + KG(s)H(s) = 0 \quad \Rightarrow \quad 1 + KP(s) = 0$$

$$P(s) = G(s)H(s) = \frac{(s + z_1)\cdots(s + z_m)}{(s + p_1)\cdots(s + p_n)} \quad ; n \geq m$$

K is a variable parameter , $KP(s)$ open loop transfer function

open loop zeros: $-z_1, -z_2, \ldots, -z_m$

open loop poles: $-p_1, -p_2, \ldots, -p_n$

Closed loop poles: roots of CLCE

Root locus: trace of closed-loop poles on s-plane

Root Locus Angle and Magnitude Conditions

If s_c is a point on the root locus, it should satisfy the characteristic equation $1+KP(s_c)=0 \Rightarrow KP(s_c)=-1$

Since $P(s_c)$ is complex, the evaluation of $KP(s_c)$ must produce $-1+j0$.

- The angle of $KP(s_c)$ must be $180° \pm k360°$ $(k=1, 2, \ldots,)$.
- The magnitude of $KP(s_c)$ must be unity.

- <u>Angle condition ($K \geq 0$) (not related to K)</u>

$$\angle P(s_c) \quad = 180° \pm k \cdot 360° \quad ; \quad k = 0, 1, 2, \cdots$$

- <u>Magnitude condition:</u> $|KP(s_c)| = 1$

Angle and Magnitude Conditions Example

$$KP(s) = K\frac{(s+z_1)}{(s+p_1)(s+p_2)}$$ s_c on root locus

Angle condition:

$$\angle P(s_1) = \angle\left\{\frac{(s_c+z_1)}{(s_c+p_1)(s_c+p_2)}\right\}$$

$$= \angle(s_c+z_1) - \angle(s_c+p_1) - \angle(s_c+p_2) = 180°$$

Magnitude condition:

$$|KP(s_c)| = \left|K\frac{(s_c+z_1)}{(s_c+p_1)(s_c+p_2)}\right|$$

$$= K\frac{|(s_c+z_1)|}{|(s_c+p_1)||(s_c+p_2)|} = 1$$

Angle and Magnitude Conditions Application

- The *angle condition* can be used to check if any particular point is on the root locus.
- The magnitude condition can be used to determine the associated value of K related to this root.

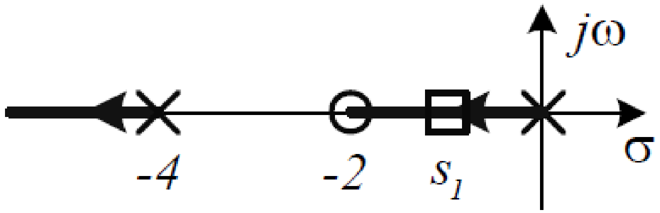
- <u>Angle condition ($K \geq 0$) (not related to K)</u>

$$\angle P(s_c) = 180^\circ \pm k \cdot 360^\circ \quad ; \quad k = 0, 1, 2, \cdots$$

- <u>Magnitude condition:</u> $|KP(s_c)| = 1$

Angle and Magnitude Conditions Example

$$KP(s) = \frac{K(s+2)}{s(s+4)}$$



Check if s_c=-1 is on the root locus.

$$P(s_c) = \frac{(-1+2)}{(-1)(-1+4)} = -\frac{1}{3}$$

Angle condition $\angle P(s_c) = \angle\left(-\frac{1}{3}\right) = 180°$ satisfied

Find the value of K which corresponds to the closed-loop root (pole) s_c=-1.

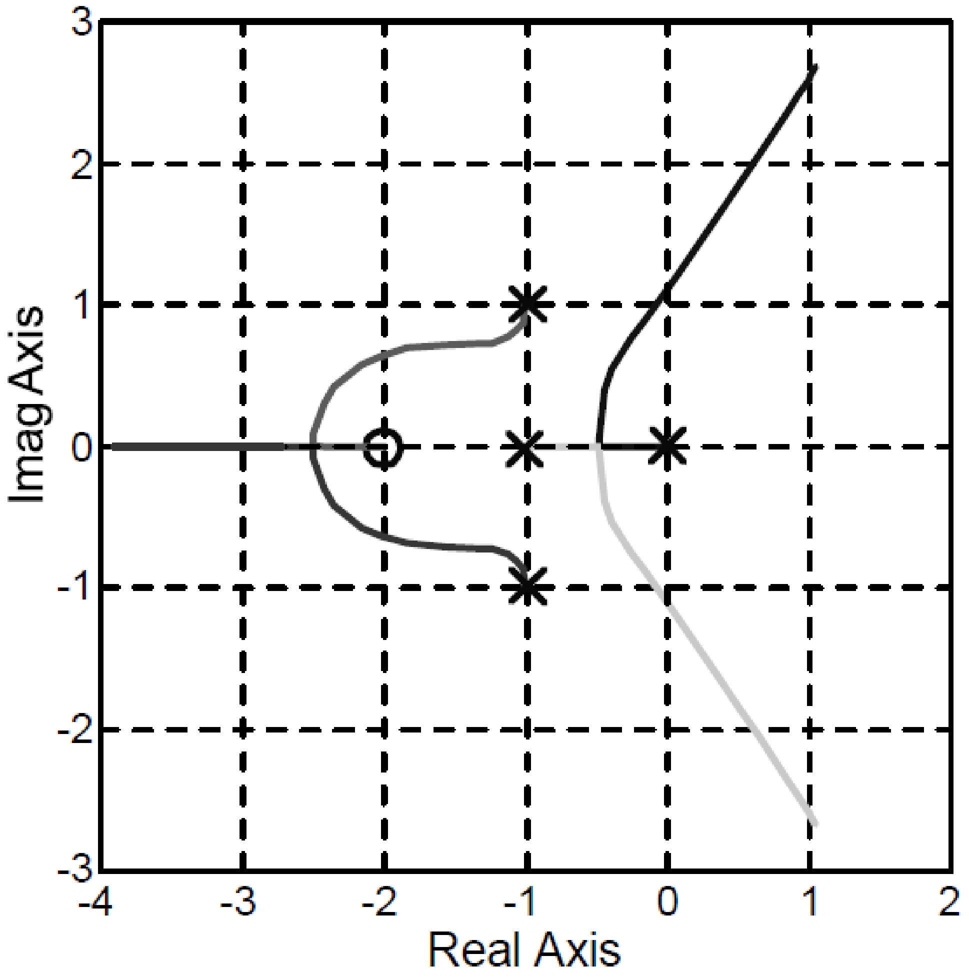
Magnitude condition $K\left|P(s_c)\right| = K\left|-\frac{1}{3}\right| = \frac{K}{3} = 1 \quad \Rightarrow \quad K = 3$

Plot Root Locus Graph Using Matlab

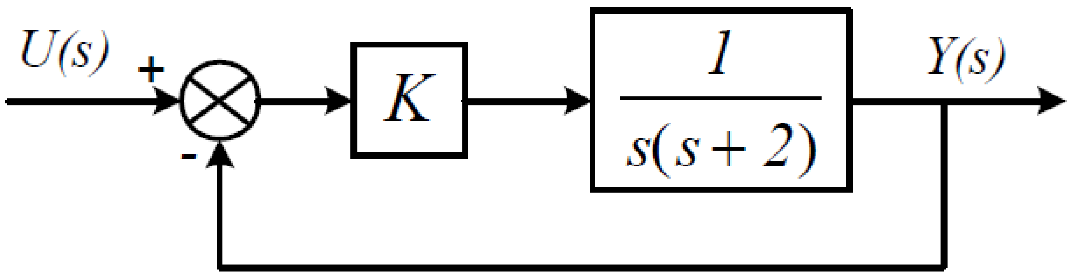
$$KP(s) = K\frac{(s+2)}{s(s+1)(s^2+2s+2)}$$

matlab command

```
z=[-2]

p=[0;-1;-1+j;-1-j]

G=zpk(z,p,1)

rlocus(G)
```



Control System Design by Parameter Adjustment



- Determine the parameter K to achieve required percentage overshoot, 15%, 10% or 5%.
- Find the settling time.

Closed-loop transfer function (CLTF):

$$T(s) = \frac{K\dfrac{1}{s(s+2)}}{1 + K\dfrac{1}{s(s+2)} \times 1} = \frac{K}{s^2 + 2s + K}$$

Control System Design by Parameter Adjustment

Closed-loop characteristic equation (CLCE):

$$s^2 + 2s + K = 0$$

2nd order standard characteristic equation (CE):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$2\zeta\omega_n = 2 \Rightarrow \zeta\omega_n = 1 \Rightarrow t_s = \frac{4}{\zeta\omega_n} = 4\,\text{sec}$$

Cannot change settling time by altering *K*

$$\left.\begin{aligned} \zeta\omega_n &= 1 \\ \omega_n^2 &= K \end{aligned}\right\} \Rightarrow \zeta = \frac{1}{\sqrt{K}} \quad or \quad K = \frac{1}{\zeta^2}$$

Control System Design by Parameter Adjustment

$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \Rightarrow \zeta = \frac{|\ln(PO)|}{\sqrt{\pi^2 + (\ln(PO))^2}}$$

$$K = \frac{1}{\zeta^2}$$

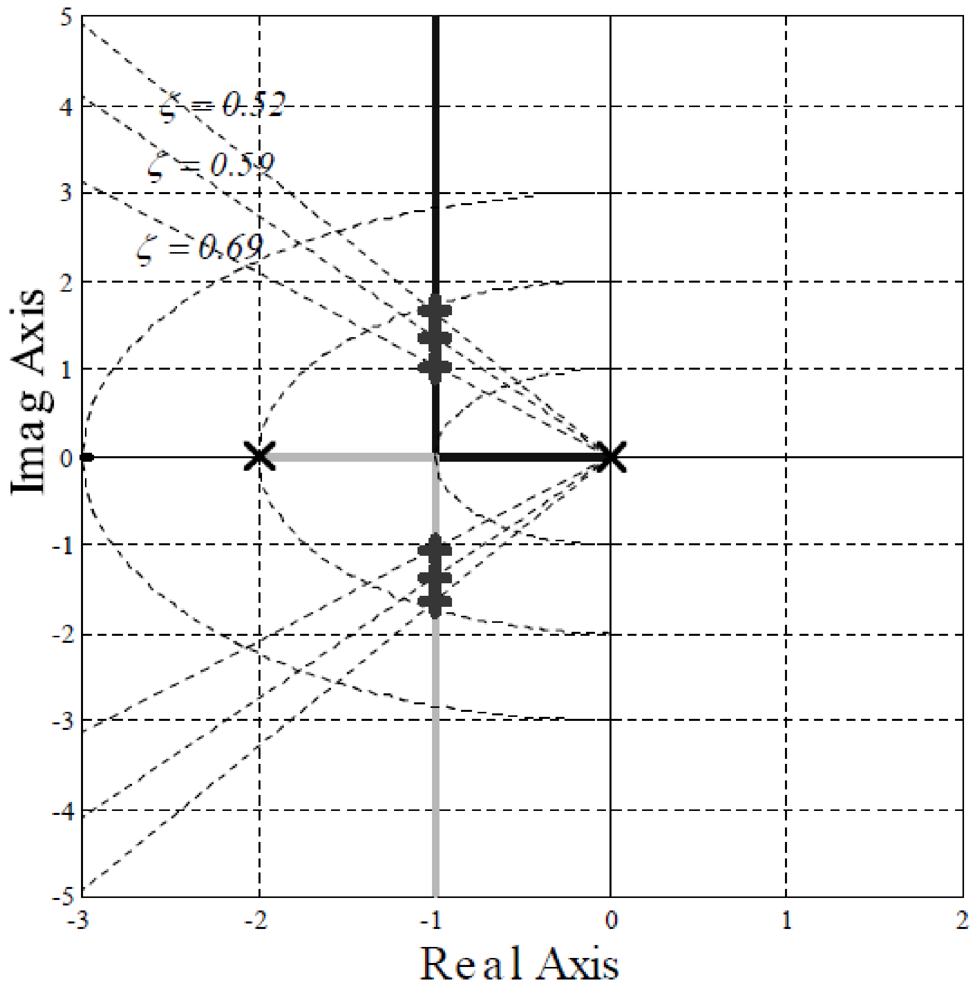
PO	ζ	K
15%	0.52	3.70
10%	0.59	2.87
5%	0.69	2.1

Design achieve by using 2nd order system formula.

Can we do design by using the root locus method?

Parameter Design Using Root Locus

```
G=tf(1,[1 2 0])
K=0:0.001:10;
rlocus(G,K)
sgrid([0.52 0.59 0.69],[1 2 3])
[K1,poles1]=rlocfind(G)
[K2,poles2]=rlocfind(G)
[K3,poles3]=rlocfind(G)
T1=feedback(K1*G,1)
T2=feedback(K2*G,1)
T3=feedback(K3*G,1)
ltiview
```



Results:

$K1=3.8$

$K2=2.8$

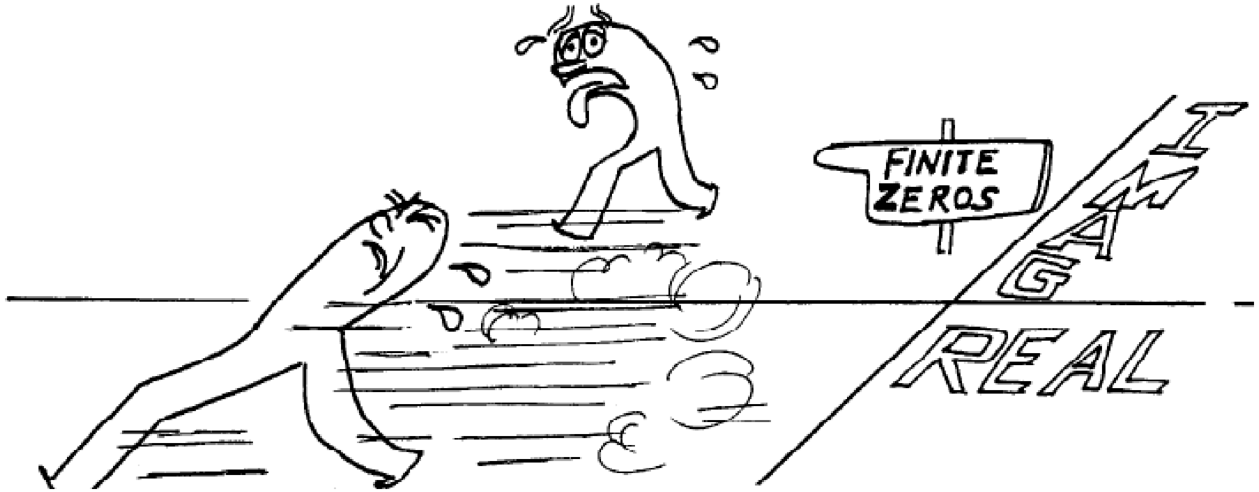
$K3=2.1$

Root Locus Property: Symmetry

- *The root locus plot is symmetric about the real axis.*
  - This is because complex roots occur in complex conjugate pairs.

$$s^2 + 2s + K = 0$$

$$s_{1,2} = -1 \pm j\sqrt{K-1}$$



"WHOA..WHOA...take it easy, man! Don't you know we'are supposed to be conjugates and get there asymptotically?"