Differential Motions of a frame

Differential motions of a frame can be divided into the following:

- Differential translations
- Differential rotations
- Differential transformations (combinations of translations and rotations)

Differential Translations

It can be represented by

$$\text{Trans}(dx, dy, dz)$$

This means that the frame has moved a differential amount along x-, y-, and z-axes.

Example

A frame B has translated a differential amount of Trans(0.01, 0.05, 0.03) units. Find its new location and orientation.

$$B = \begin{bmatrix} 0.707 & 0 & -0.707 & 5 \\ 0 & 1 & 0 & 4 \\ 0.707 & 0 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The new location of the frame is:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0.01 \\ 0 & 1 & 0 & 0.05 \\ 0 & 0 & 1 & 0.03 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times B$$

$$= \begin{bmatrix} 0.707 & 0 & -0.707 & 5.01 \\ 0 & 1 & 0 & 4.05 \\ 0.707 & 0 & 0.707 & 9.03 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Rotations about the Reference Axes

A differential rotation is a small rotation of the frame. It is generally represented by $Rot(q, d\theta)$, which means that the frame has rotated an angle of $d\theta$ about an axis q.

Specifically, differential rotations about the x-, y-, and z-axis are defined by

δx, δy, and δz.

Since the rotations are small amounts, we can use the following approximations:

$$\begin{cases} \sin \delta x = \delta x & \text{(in radians)} \\ \cos \delta x = 1 \end{cases}$$

Then, the rotation matrices representing differential rotations about x-, y-, and z-axes will be:

$$Rot(x, \delta x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(y, \delta y) = \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(z, \delta z) = \begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly, we can also define differential rotations about the current axes are:

$$Rot(n, \delta n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta n & 0 \\ 0 & \delta n & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(o, \delta o) = \begin{bmatrix} 1 & 0 & -\delta o & 0 \\ 0 & 1 & 0 & 0 \\ -\delta o & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(a, \delta a) = \begin{bmatrix} 1 & -\delta a & 0 & 0 \\ \delta a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\sqrt{1^2 + \delta a^2} > 1 \quad \simeq 1}$$

$\delta a \, \delta o \qquad \delta a \, \delta a$

$0.01 \times 0.01 = 0.0001$

$\delta a \qquad \delta a^2 \leftarrow$ ignore δa^2 compared to δa

In mathematics, higher-order differentials are considered negligible and are usually neglected.
If we do neglect the higher-order differentials such as $(\delta x)^2$, the magnitude of the vectors remain acceptable.

$$Rot(x, \theta)\,Rot(y, \alpha) \neq Rot(y, \alpha)\,Rot(x, \theta)$$

Chapter 2 : Not Eqaul

If we multiply two differential motions in different orders, we get :

$$Rot(x, \delta x)\, Rot(y, \delta y) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \delta y & 0 \\ \delta x \delta y & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(y, \delta y)\, Rot(x, \delta x) =$$

$$\begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & \delta x \delta y & \delta y & 0 \\ 0 & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If we set higher-order differentials such as $\delta x \delta y$ to zero, the results are exactly the same.

Consequently, for differential motions, the order of multiplication is no longer important and

$$Rot(x, \delta x) Rot(y, \delta y) = Rot(y, \delta y) Rot(x, \delta x)$$

Differential Rotation about a General Axis q

we can assume that a differential rotation about a general axis q is composed of three differential rotations about the three axes, in any order,

or $(d\theta)\vec{q} = (\delta x)\vec{i} + (\delta y)\vec{j} + (\delta z)\vec{k}$

$$Rot(\vec{q}, d\theta) = Rot(x, \delta x)\, Rot(y, \delta y)\, Rot(z, \delta z)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta x \delta y + \delta z & -\delta x \delta y \delta z + 1 & -\delta x & 0 \\ -\delta y + \delta x \delta z & \delta x + \delta y \delta z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

IF we nglect all higher-order differentials:

$$Rot(\vec{q}, d\theta) = \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Important

Example

Find the total differential transformation caused by small rotations about the three axes of $\delta x = 0.1$, $\delta y = 0.05$, $\delta z = 0.02$ radians.

Substituting the given rotations in equation

$$\begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -0.02 & 0.05 & 0 \\ 0.02 & 1 & -0.1 & 0 \\ -0.05 & 0.1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next week

Differential Transformations

of frame