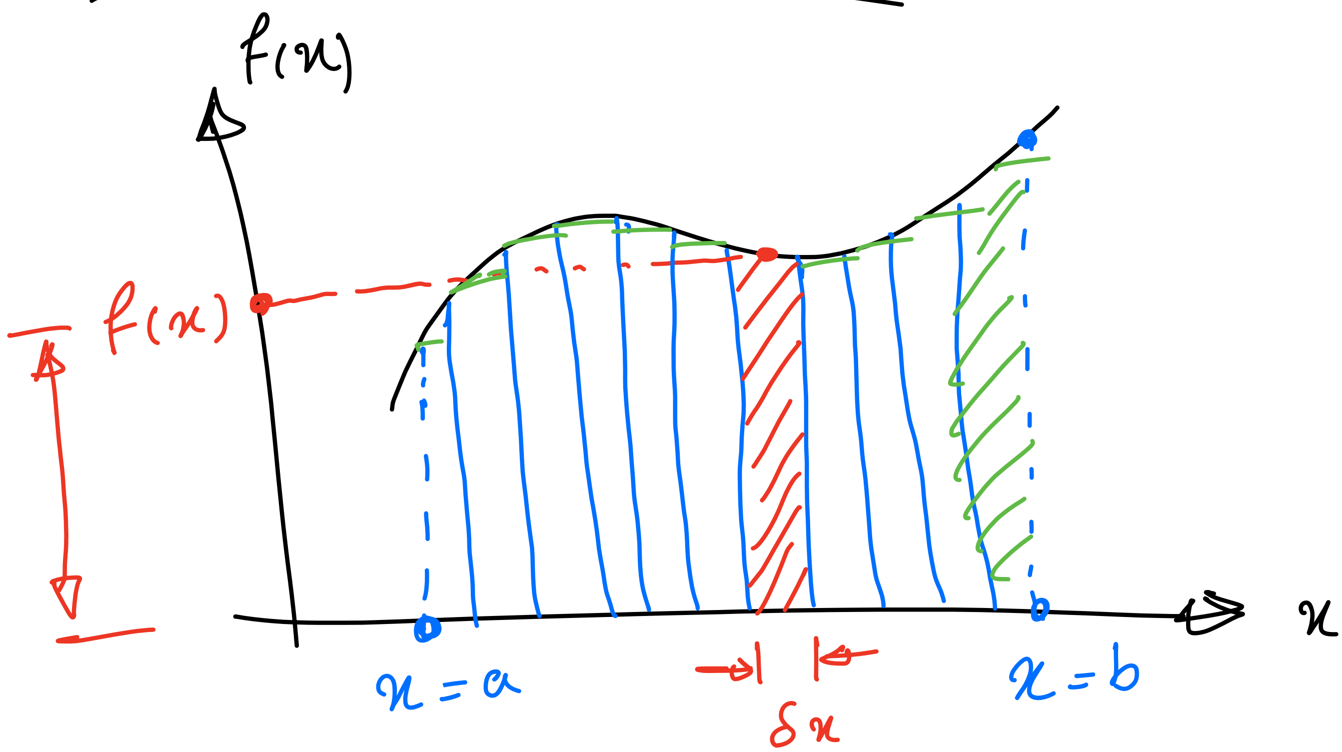
Integration



The area of the rectangle is

$$\delta A = f(x)\,\delta x$$

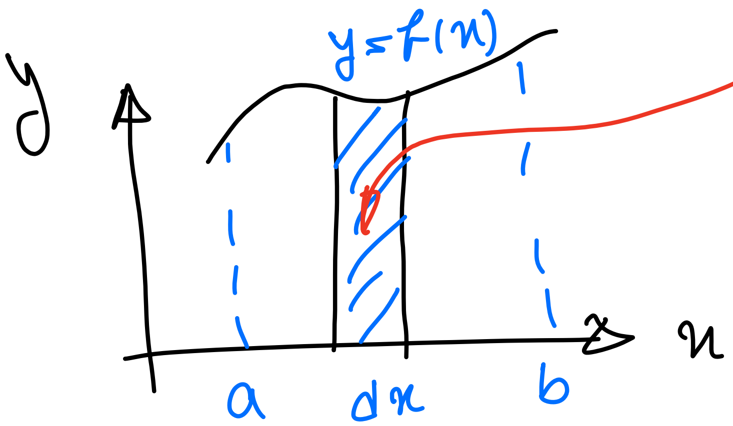
Application: Area under the curve

In the limit when $\delta x \longrightarrow 0$

the area is defined by the integral:

$$\int_a^b f(x)\,dx$$

The differential element



$$dA = f(x)\,dx$$

$$\int_a^b dA = \int_a^b f(x)\,dx$$

$$A = \int_a^b f(x)\,dx$$

Indifinite integral

$$F(x) = \int f(x)\,dx$$

if a and b limits are not given

For the definit integral:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$F = \int f(x)\,dx$$

For indefinite integral:

$$\int f(x)\,dx = F(x) + \underset{\substack{\downarrow \\ \text{constant}}}{c}$$

For example:

$$\int \sin x\,dx = -\cos x + \underset{\substack{\downarrow \\ \text{constant}}}{c}$$

$$\int_0^{90^\circ} \sin x\,dx = -\cos x\Big]_0^{90}$$

$$= -\cos(90^\circ) - (-\cos 0)$$

$$= 1$$

Definite

Fundumental thearm of calculus

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

Example

$$\int \sin 3x\,dx = ?$$

$3x = u$ chang of variable

$\Longrightarrow 3\,dx = du$ or $dx = \frac{du}{3}$

substitute:

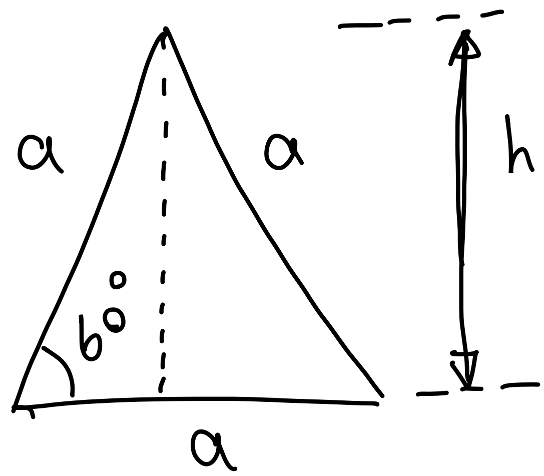
$$\int \sin u \frac{du}{3} = \frac{1}{3}\int \sin u\,du$$

$$= -\frac{1}{3}\cos u + C$$

$(u = 3x) \rightarrow \quad = -\frac{1}{3}\cos 3x + C$

Example

Calculating area of a triangle:



we can calculate area by:

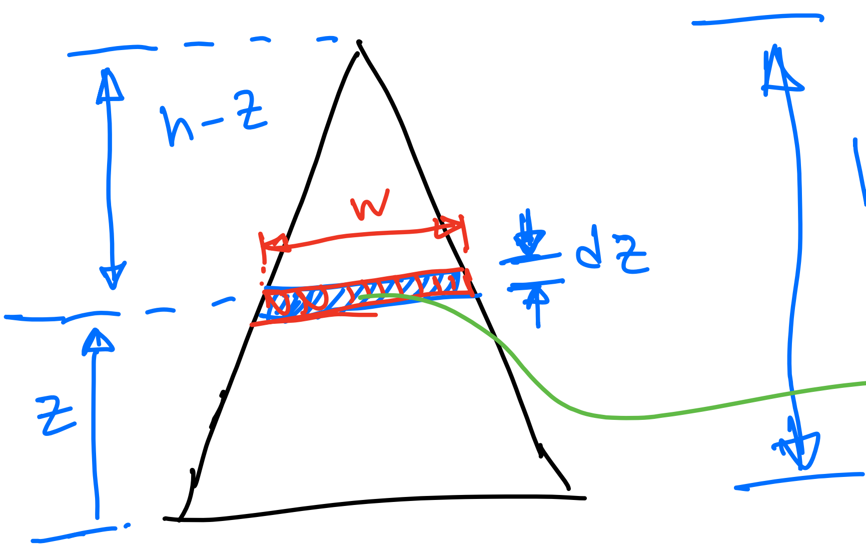
$$\text{Area} = \frac{1}{2} a h$$

$$h = a \sin 60^\circ$$

$$h = a \sqrt{3}/2$$

$$\text{Area} = \frac{\sqrt{3}}{4} a^2 \quad *$$

Now solve the problem using the integral approach:



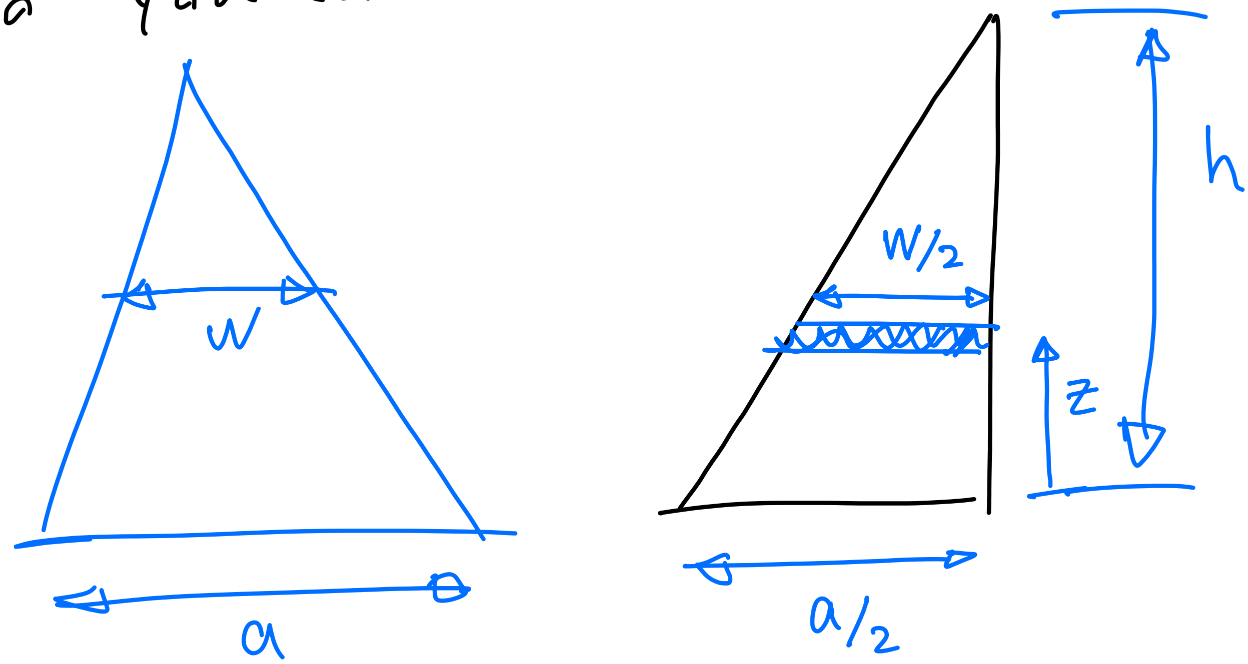
$$h = a \frac{\sqrt{3}}{2}$$

$$dA = w\,dz$$

$$\int dA = \int w \, dz$$

$$A = \int_0^h w \, dz$$

Find w as a function of z so you can calculate the integral



$$\frac{w/2}{z} = \frac{a/2}{h} \Rightarrow w = \frac{az}{h}$$

substitute in the integral:

$$A = \int_0^h w\,dz = \int_0^h \frac{a z}{h}\,dz$$

$$A = \frac{a}{h}\,\frac{z^2}{2}\Big|_0^h$$

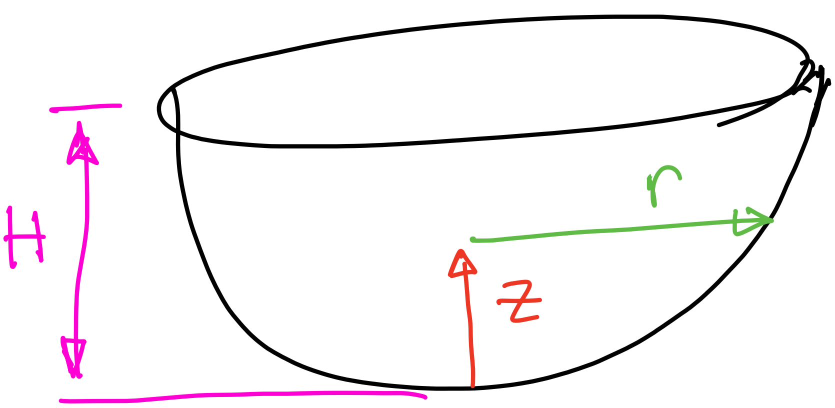
$$A = \frac{a}{h}\,\frac{h^2}{2}$$

$$h = \frac{\sqrt{3}}{2}a \longrightarrow A = \frac{\sqrt{3}}{4}a^2$$

The same resut as (*)

Example

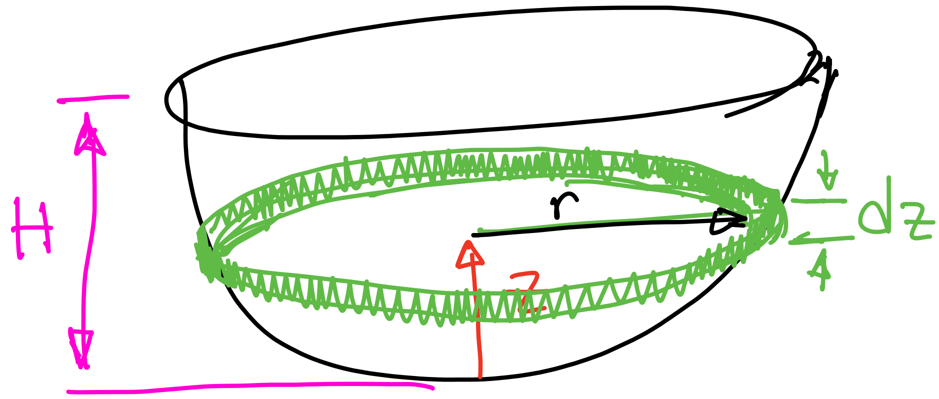
Find the volume of the cup.



$r(z) = \sqrt{kz}$

k is a constant

k is a given value



The differential element is a disk with a thickness of dz

The volume of the disk is:

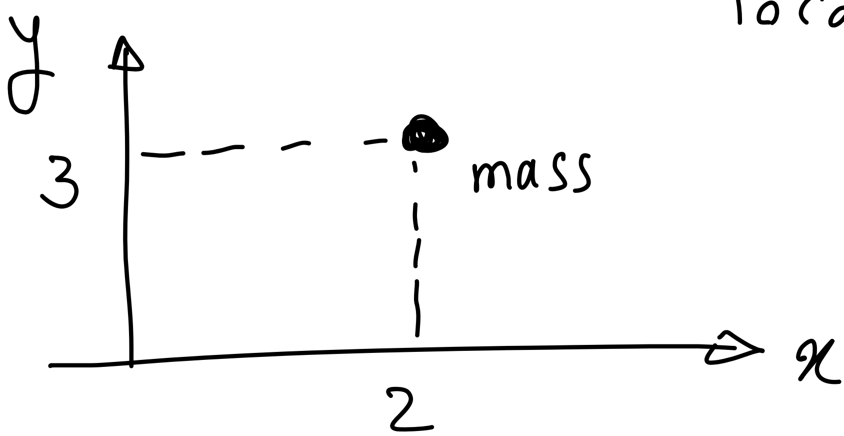
$$dV = \pi r^2 dz \qquad r = \sqrt{Kz}$$

$$dV = \pi Kz\,dz$$

$$\int_0^H dV = \int_0^H \pi Kz\,dz$$

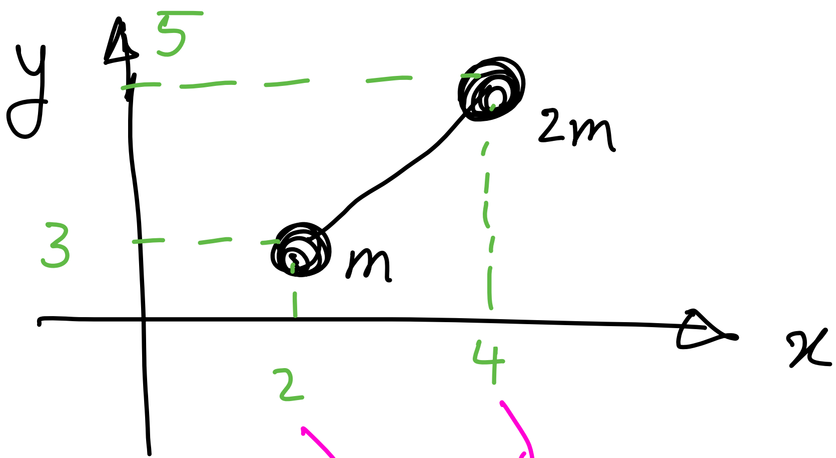
$$V = \pi K \left.\frac{z^2}{2}\right]_0^H = \pi K \frac{H^2}{2}$$

Another application is in finding the center of mass / centroid:



location of the mass

$2\hat{i} + 3\hat{j}$



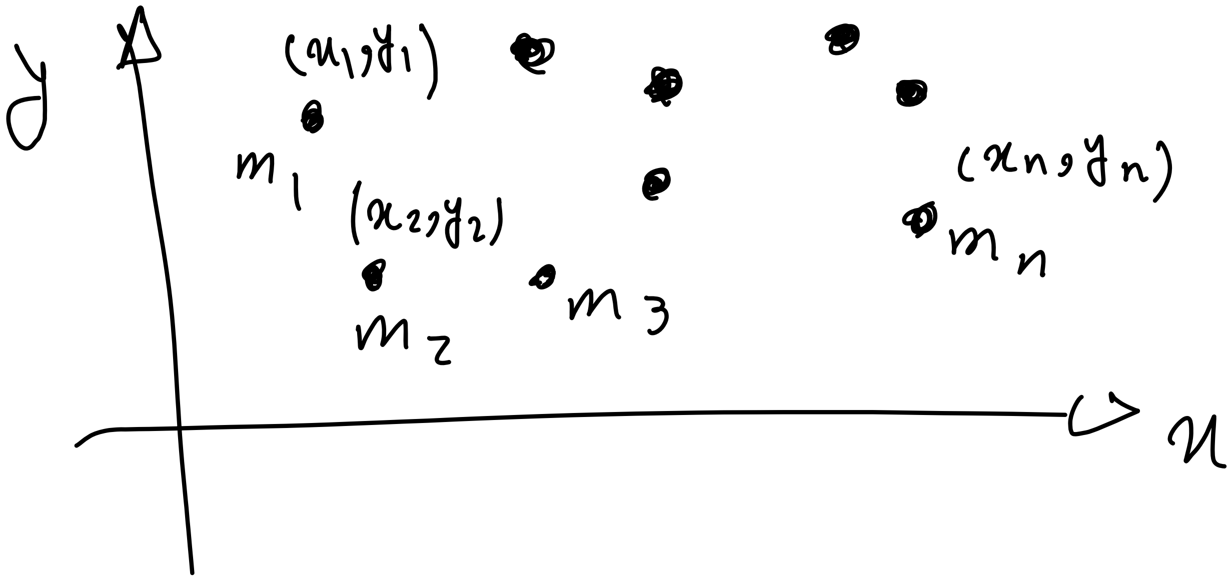
Total mass = 3m

Find the center of mass:

$$X_{cm} = \frac{(2)m + 4(2m)}{3m} = \frac{10}{3}$$

$$Y_{cm} = \frac{3(m) + 5(2m)}{3m} = \frac{13}{3}$$

More General



$$m_{Total} = m_1 + m_2 + m_3 \cdots + m_n$$

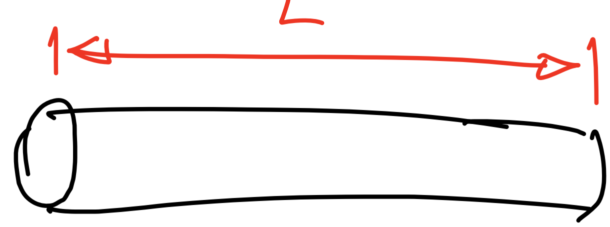
$$X_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

$\sum m_i =$ Total mass

$$Y_{cm} = \frac{\sum m_i y_i}{\sum m_i}$$

Example:

Rod with a raduis of a.



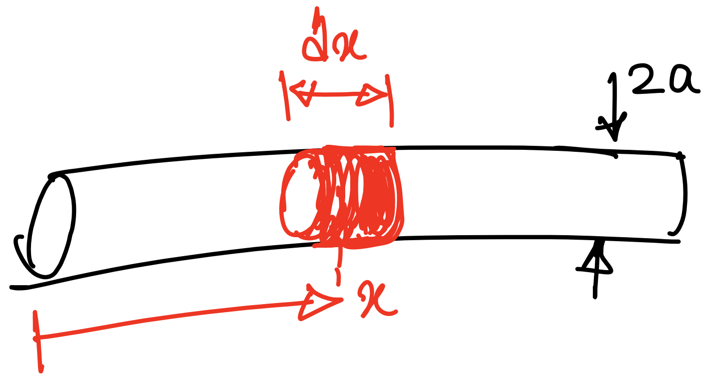
Desity $\rho \rightarrow$ constant

Mass $= m = \pi a^2 L \rho$ $\quad \boxed{m = \rho V}$

$X_{cm} = \frac{1}{2} L \longleftrightarrow$ we already know

Integral approach:

$$X_{cm} = \frac{\sum m_i x_i}{\sum m_i} \longrightarrow \frac{\int x \, dm}{\text{Total mass}}$$



volum $= \pi a^2 dx$

$$X_{cm} = \frac{\int x \, dm}{\text{total } m} = \frac{\int x \rho \pi a^2 dx}{\text{total } m}$$

$$X_{cm} = \frac{\rho \pi a^2 \int_0^L x \, dx}{\rho \pi a^2 L} = \frac{L}{2}$$