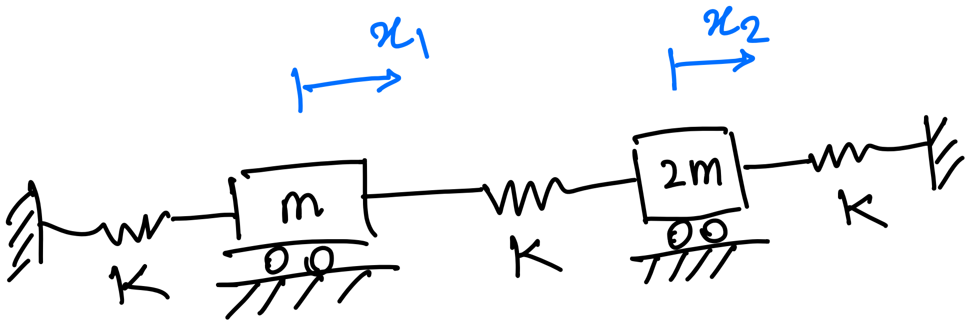
Chapter 5

Two Degrees of Freedom (2DOF)

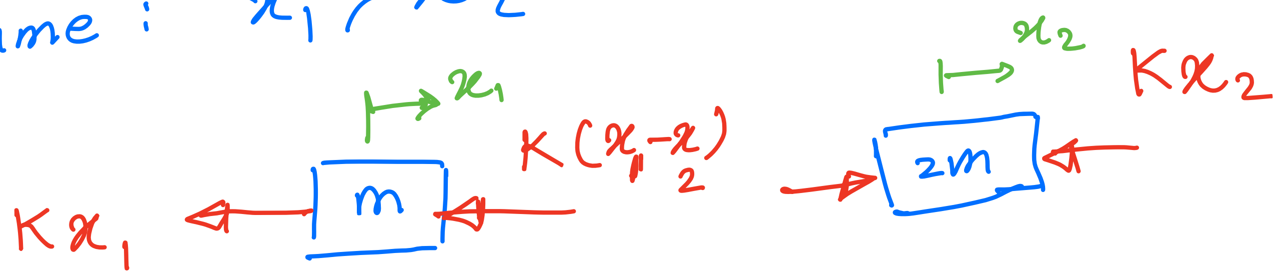
5.1

The normal mode Analysis



F.B.D.

Assume : $x_1 > x_2$



$$\begin{cases} \overset{+}{\rightarrow}\Sigma F = m\ddot{x}_1 \Rightarrow -Kx_1 - K(x_1-x_2) = m\ddot{x}_1 \\ \overset{+}{\rightarrow}\Sigma F = 2m\ddot{x}_2 \Rightarrow K(x_1-x_2) - Kx_2 = 2m\ddot{x}_2 \end{cases}$$

Assume:

$$\begin{cases} x_1 = A_1 \sin\omega t & \text{or} \quad A_1 e^{i\omega t} \\ x_2 = A_2 \sin\omega t & \text{or} \quad A_2 e^{i\omega t} \end{cases}$$

$$\begin{cases} \ddot{x}_1 = -A_1 \omega^2 \sin\omega t \\ \ddot{x}_2 = -A_2 \omega^2 \sin\omega t \end{cases}$$

substitute the solution in the equations of motion

$$(2K - \omega^2 m) A_1 - K A_2 = 0 \qquad (1)$$

$$-K A_1 + (2K - 2\omega^2 m) A_2 = 0 \qquad (2)$$

$$\begin{bmatrix} 2K - \omega^2 m & -K \\ -K & 2K - 2\omega^2 m \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\omega^2 = \lambda$

$$\begin{vmatrix} 2K - \lambda m & -K \\ -K & 2K - 2\lambda m \end{vmatrix} = 0$$

$$(2k-\lambda m)(2k-2\lambda m)-k^2=0$$

$$\lambda^2-\left(3\frac{k}{m}\right)\lambda+\frac{3}{2}\left(\frac{k}{m}\right)^2=0$$

The roots λ_1 and λ_2 are:

$$\lambda_1=\left(\frac{3}{2}-\frac{1}{2}\sqrt{3}\right)\frac{k}{m}=0.634\frac{k}{m}$$

$$\lambda_2=\left(\frac{3}{2}+\frac{1}{2}\sqrt{3}\right)\frac{k}{m}=2.366\frac{k}{m}$$

λ_1 and λ_2 are the eigenvalues of the system.

The natural frequencies of the system are

$$w_1=\sqrt{\lambda_1}=\sqrt{0.634\frac{k}{m}}$$

$$w_2=\sqrt{\lambda_2}=\sqrt{2.366\frac{k}{m}}$$

From equation 1 and 2, the ratio of the amplitudes:

$$\frac{A_1}{A_2} = \frac{K}{2K - w^2 m} = \frac{2K - 2w^2 m}{K}$$

substituting the natural frequencies w_1 and w_2:

For $w_1^2 = 0.634 \frac{K}{m}$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{K}{2K - w_1^2 m} = \frac{1}{2 - 0.634} = 0.731$$

similarly:

$$w_2^2 = 2.365 \frac{K}{m}$$

$$\left(\frac{A_1}{A_2}\right)^{2} = \frac{K}{2K - w_2^2 m} = \frac{1}{2 - 2.366} = -2.73$$

<u>Normal modes</u> (Normalized)

we can chose one amplitude to be equal to 1.

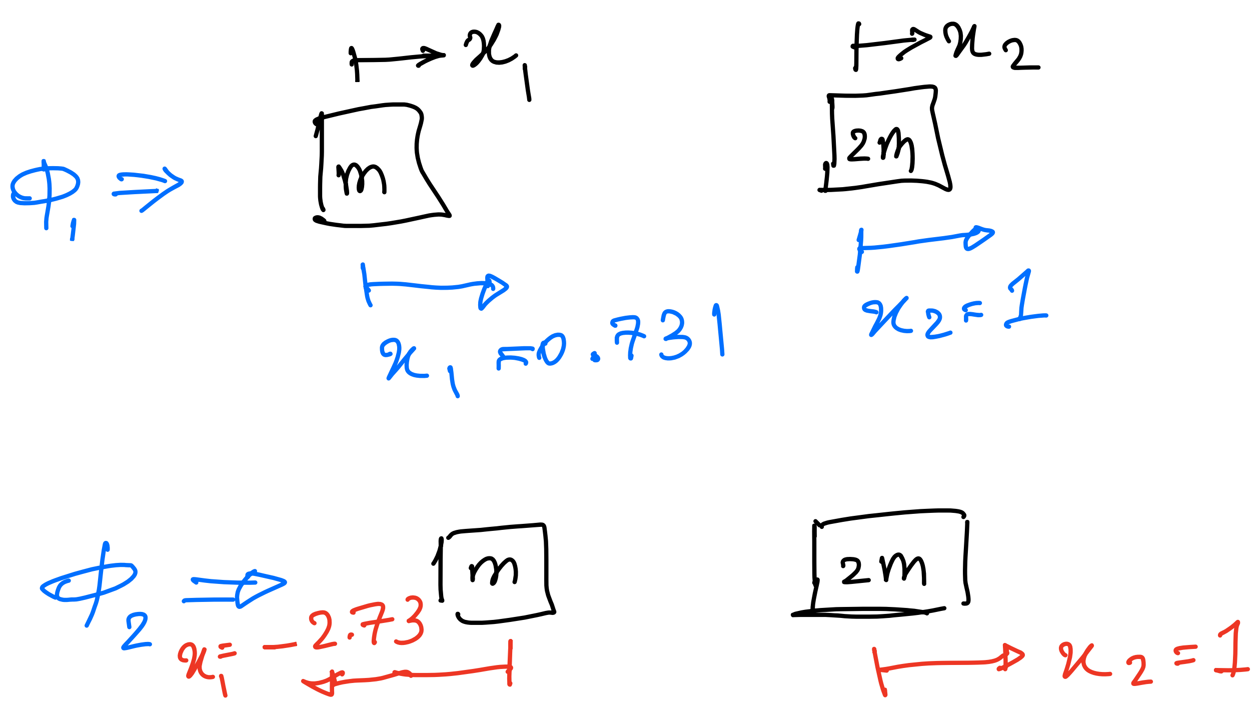
Assume $A_2 = 1$

The two normal modes, which we call eigenvectors (in vibrations we call them mode shapes)

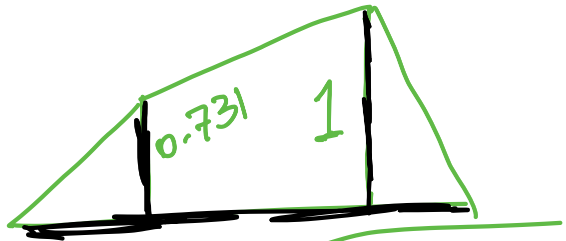
$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \qquad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix}$$

mode shape 1 corresponding to w_1

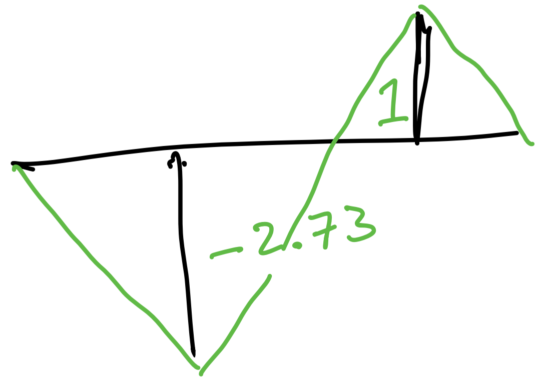
corresponding to w_2



show the normal modes (mode shapes)



$$\omega_1 = \sqrt{0.634 \frac{K}{m}}$$



Each normal mode can then be written as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(1)} = A_1 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \psi_1) \quad \text{Eq. } 3$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(2)} = A_2 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2) \quad \text{Eq. } 4$$

5.2 Initial conditions

Initial dirplacements $x_1(0)$, $x_2(0)$

Initial velocities $\dot{x}_1(0)$, $\dot{x}_2(0)$

Example (5.2-1)

Determine the free vibration for the 2DOF system above for initial canditions:

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix} \qquad \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

substituting these initial canditions into equations 3 and 4 gives:

$$\begin{Bmatrix} 2 \\ 4 \end{Bmatrix} = A_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin \psi_1 + A_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \sin \psi_2$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 A_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos \psi_1 + \omega_2 A_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos \psi_2$$

4 equations and 4 unknowns.

Example 5.2-1 $\rightarrow$

$A_1 = 3.732$

$A_2 = 0.268$

Therefore the solution is

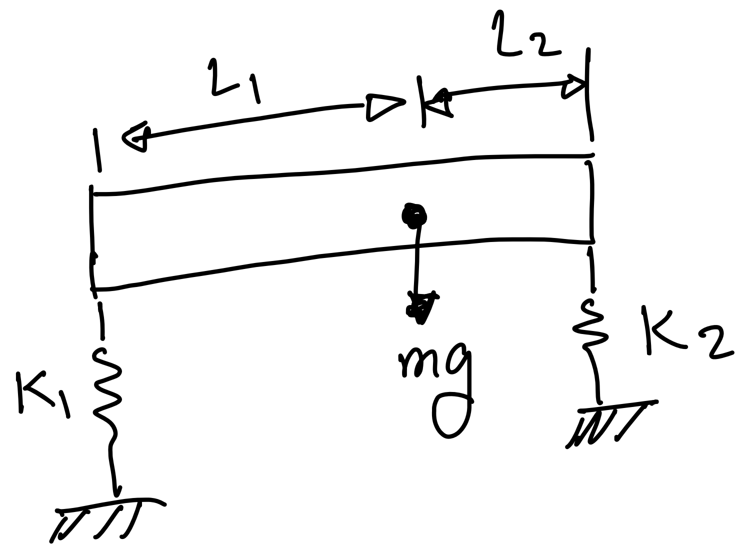
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 3.732 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos \omega_1 t$$

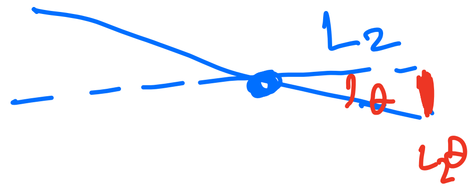
$$+ 0.268 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos \omega_2 t$$

$$= \begin{Bmatrix} 2.732 \\ 3.732 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -0.732 \\ 0.268 \end{Bmatrix} \cos \omega_2 t$$

Example 5.3-1

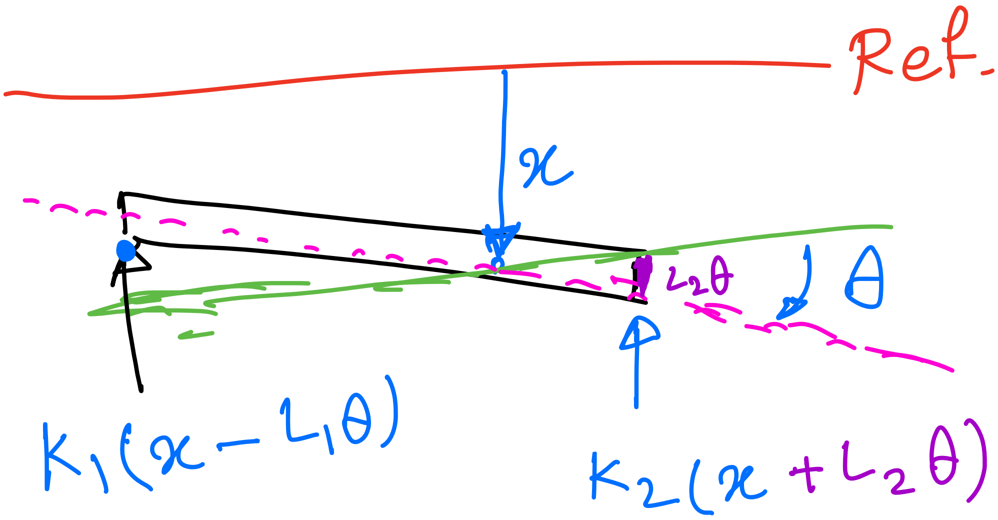
A half vehicle model (2DOF model)





The DoF. are one traslation and one rotation.

F.B.D.



Equations of motion:

$$\overset{+}{\downarrow}\ \Sigma F = m\ddot{x}$$

$$\overset{+}{\circlearrowright}\ \Sigma M_{cg} = J\ddot{\theta}$$

J is the moment of inertia

$$\begin{cases} -K_1(x-L_1\theta) - K_2(x+L_2\theta) = m\ddot{x} \\ -K_2(x+L_2\theta)L_2 + K_1(x-L_1\theta)L_1 = J\ddot{\theta} \end{cases}$$

$$\begin{cases} m\ddot{x} + x(+K_1+K_2) + \theta(-L_1K_1+L_2K_2) = 0 \\ J\ddot{\theta} + x(K_2L_2 - K_1L_1) + \theta(K_2L_2^2 + K_1L_1^2) = 0 \end{cases}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} K_1+K_2 & -L_1K_1+L_2K_2 \\ K_2L_2-K_1L_1 & K_2L_2^2+K_1L_1^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = 0$$

Example 5.3-2

2 DoF

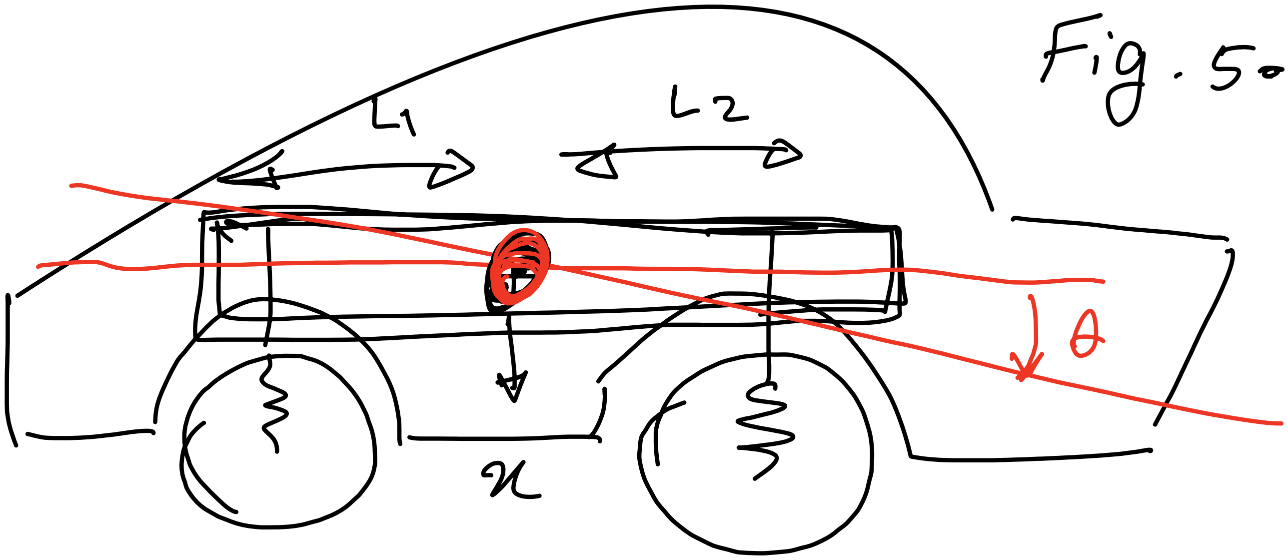
$W = 3220$ lb $\quad l_1 = 4.5$ ft

$J = \frac{W}{g} r^2 \quad L_2 = 5.5$ ft $\quad K_2 = 2600$

$K_1 = 2400 \frac{\text{lb}}{\text{ft}}$

$r = 4$ ft $\quad L = 10$ ft

Fig. 5-3-5



F.B.D.
Important
↓
Eq. Mation

$$\begin{cases} x = X \sin \omega t \\ \theta = \Theta \sin \omega t \end{cases}$$

↓ Substitute in Eq.

[Matrix form]

↓

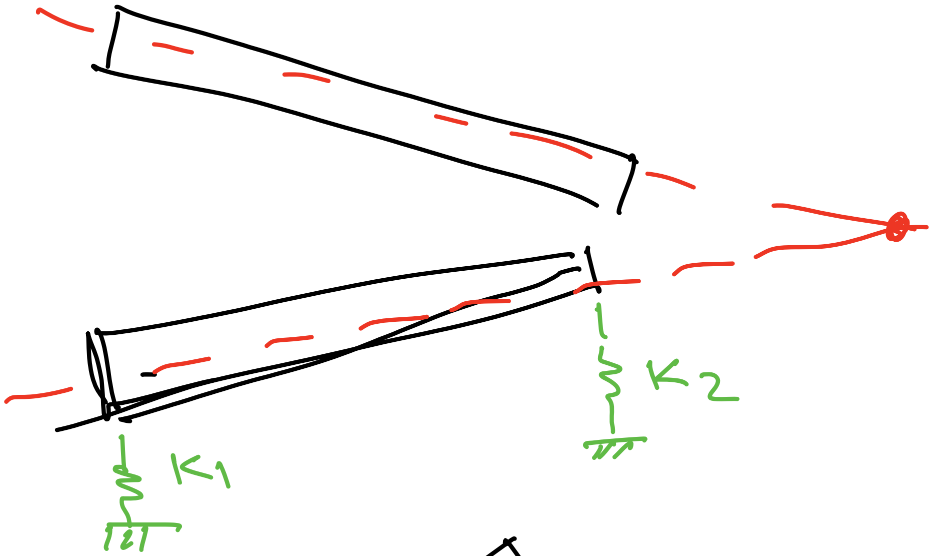
eigen values

↓

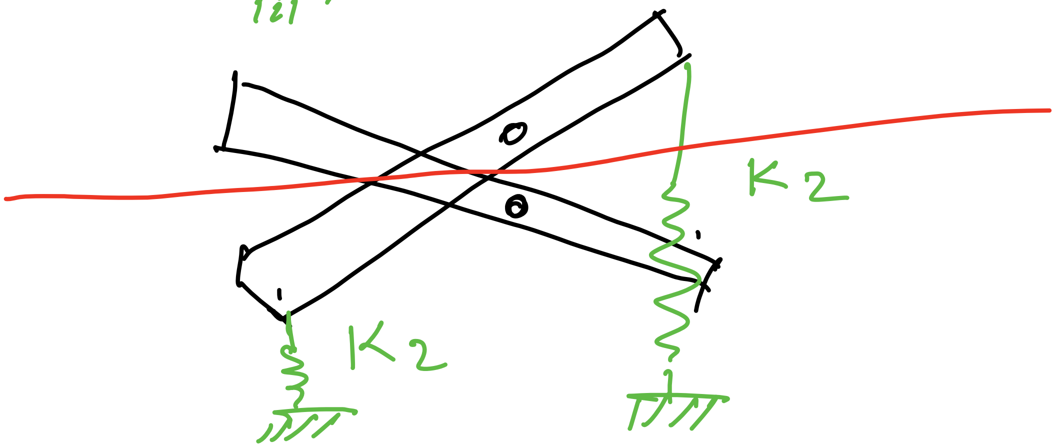
eigen vectors

↓

Made shapes



Mode shape 1



Mode shape 2