Chapter 3

Differential Motions and velocities

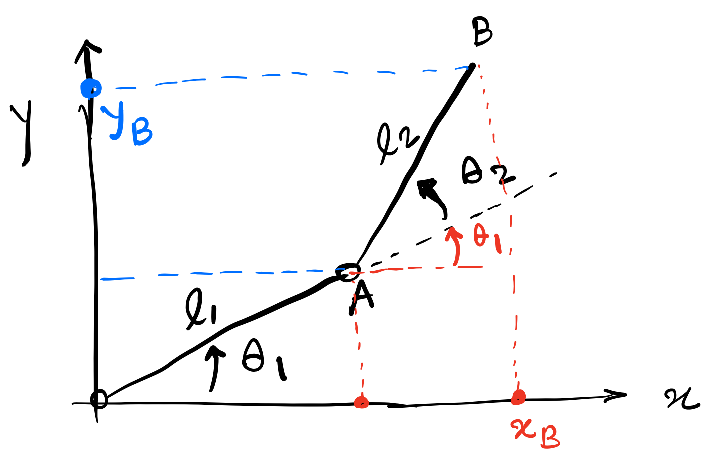
Differential motions are small movements of mechanisms (e.g. robots) that can be used to derive velocity relationships between different parts of the mechanism.

position $\to x$ small movement $\to \Delta x$

velocity $= \frac{\Delta x}{\Delta t}$

Differential Relationships

Example: 2-DOF mechanism



$$\begin{cases} x_B = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) \\ y_B = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1+\theta_2) \end{cases}$$

$$\text{velocity} = \frac{d\,(\text{Displacement})}{dt}$$

$$\begin{cases} \dfrac{dx_B}{dt} = \ell_1 \dfrac{d(\cos\theta_1)}{dt} + \ell_2 \dfrac{d(\cos(\theta_1+\theta_2))}{dt} \\[2ex] \dfrac{dy_B}{dt} = \ell_1 \dfrac{d(\sin\theta_1)}{dt} + \ell_2 \dfrac{d(\sin(\theta_1+\theta_2))}{dt} \end{cases}$$

$$\frac{d(\cos\theta)}{dt} = \frac{d(\cos\theta)}{dt}\,\frac{d\theta}{d\theta} = \frac{d\cos\theta}{d\theta}\,\frac{d\theta}{dt} = (-\sin\theta)\,\dot{\theta}$$

$$\begin{cases} \dfrac{dx_B}{dt} = \ell_1 (-\sin\theta_1)\,\dot{\theta}_1 - \ell_2 (-\sin(\theta_1+\theta_2))(\dot{\theta}_1+\dot{\theta}_2) \\[2ex] \dfrac{dy_B}{dt} = \ell_1 (\cos\theta_1)\,\dot{\theta}_1 + \ell_2 (\cos(\theta_1+\theta_2))(\dot{\theta}_1+\dot{\theta}_2) \end{cases}$$

$$\left(\dot{\theta} = \frac{d\theta}{dt}\right)$$

$$\begin{bmatrix} dx_B \\ dy_B \end{bmatrix} \Big/ dt = \underbrace{\begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix} \Big/ dt$$

Differential motion of B

Differential motion of joints

Jacobian

The Jacobian is a representation of the geometry of the elements of a mechanism in time.

Suppose we have a set of equations y_i in terms of a set of variables x_j as :

$$y_i = f_i(x_1, x_2, x_3, \dots, x_j)$$

$\theta_1, \theta_2 \dots$

Forward Kinematic equations (x, y, z, orientations)

The differential change in y_i as a result of a differential change in x_j will be:

$$\left\{\begin{array}{l}
\delta y_1 = \dfrac{\partial f_1}{\partial x_1}\delta x_1 + \dfrac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \dfrac{\partial f_1}{\partial x_j}\delta x_j \\[2ex]
\delta y_2 = \dfrac{\partial f_2}{\partial x_1}\delta x_1 + \dfrac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \dfrac{\partial f_2}{\partial x_j}\delta j \\[2ex]
\vdots \\[2ex]
\delta y_i = \dfrac{\partial f_i}{\partial x_1}\delta x_1 + \dfrac{\partial f_i}{\partial x_2}\delta x_2 + \cdots + \dfrac{\partial f_i}{\partial x_j}\delta x_j
\end{array}\right.$$

$$\begin{bmatrix} \delta y_1 \\ \delta y_2 \\ \vdots \\ \delta y_i \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & & \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \cdots \\ \vdots & & & \\ \frac{\partial f_i}{\partial x_1} & \cdots & & \frac{\partial f_i}{\partial x_j} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_j \end{bmatrix}$$

$$[\delta y_i] = \left[\frac{\partial f_i}{\partial x_j} \right] [\delta x_j]$$

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \text{Robot} \\ \text{Jacobian} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

$$[D] = [J][D_\theta]$$

dx, dy, and dz in $[D]$ represent the differential motions of the hand along x-, y-, and z-axes.

δx, δy, and δz in [D] represent the differential rotations of the hand around x-, y-, and z,

$[D_\theta]$ represents the differential motions of the joints.

If [D] and $[D_\theta]$ are divided by dt, they will represent velocities instead of differential motions.

In this chapter, we will work with the differential motions rather than velocities, knowing that in all relationships, by simply dividing the differential motions by dt we can get the velocities.

Example

The Jacobian of a robot at a particular time is given. Calculate the linear and angular differential motions of the robot's hand frame for the given joint differential motions.

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_\theta = \begin{bmatrix} 0 \\ 0.1 \\ -0.1 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}$$

$$[D] = [J][D_\theta] = \begin{bmatrix} 0 \\ -0.1 \\ 0.1 \\ 0 \\ -0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$