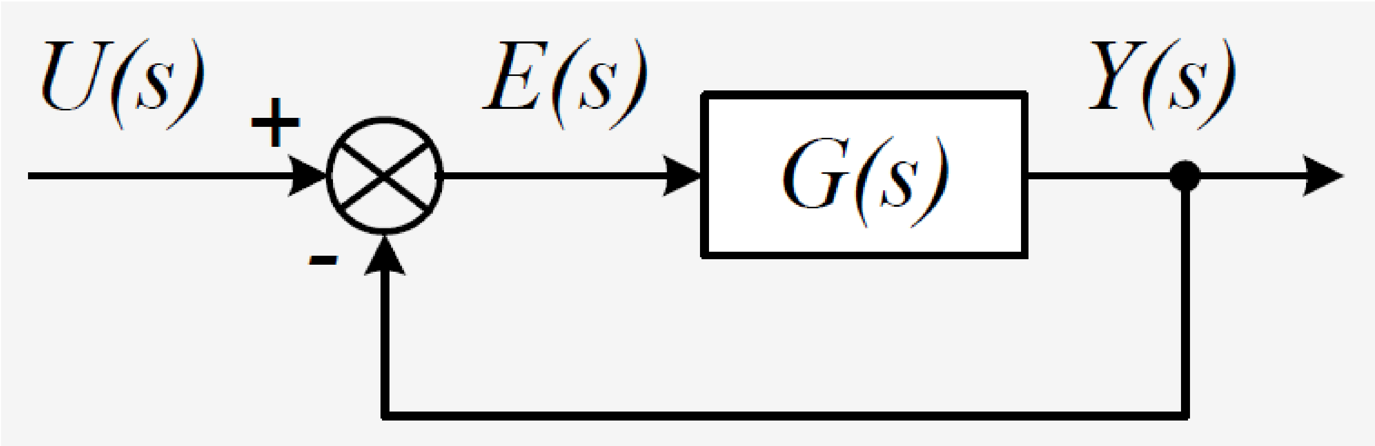
Instrumentation and Controls
ETM 3301

Lecture 16

Instructor

Dr. Farbod Khoshnoud

Unity negative feedback systems



- Unity feedback:
  - The TF of the feedback path is *1*.
  - The output is directly feedback into the comparison unit.

G(s) forward path TF *U(s)* input signal

Y(s) output signal *E(s)* error signal

Unity negative feedback system error

- The closed-loop TF is:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)}$$

- The error signal is:

$$E(s) = U(s) - Y(s) = U(s) - T(s)U(s)$$

$$= [1 - T(s)]U(s) = \left[1 - \frac{G(s)}{1 + G(s)} \right] U(s)$$

$$= \frac{1}{1 + G(s)} U(s)$$

Unity negative feedback system steady state error

- The steady state error can be found by using the Final Value Theorem:

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sU(s)}{1 + G(s)}$$

- Note:
  - This theorem is only valid when e_{ss} exists.
  - If the closed-loop TF *T(s)* is stable, the steady state error e_{ss} exists.
  - e_{ss} is evaluated using test signals.

Unity negative feedback system steady state error *for unit step input*

$$u(t) = 1 \qquad U(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s \to 0} \frac{sU(s)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

- In order to have zero steady-state error for unit step input, we must have: $\lim_{s \to 0} G(s) = \infty$
- The ***position error constant***: $K_p = \lim_{s \to 0} G(s)$
  - The dc gain of the forward path (open-loop) TF.

$$e_{ss} = \frac{1}{1 + K_p}$$

Unity negative feedback system steady state error *for unit ramp input*

$$u(t) = t \qquad U(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s\to 0} \frac{sU(s)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim\limits_{s\to 0} sG(s)}$$

- In order to have zero steady-state error for ramp input, we must have: $\lim\limits_{s\to 0} sG(s) = \infty$
- The ***velocity error constant***: $K_v = \lim\limits_{s\to 0} sG(s)$

$$e_{ss} = \frac{1}{K_v}$$

Unity negative feedback system steady state error *for unit parabolic input*

$$u(t) = \frac{1}{2}t^2; \quad U(s) = \frac{1}{s^3}$$

$$e_{ss} = \lim_{s \to 0} \frac{sU(s)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- In order to have zero steady-state error for parabolic input, we must have: $\lim_{s \to 0} s^2 G(s) = \infty$
- The ***acceleration error constant***: $K_a = \lim_{s \to 0} s^2 G(s)$

$$e_{ss} = \frac{1}{K_a}$$

Steady state errors of unity negative feedback systems

Input $u(t)$	e_{ss}
Unit Step 1	$\dfrac{1}{1+K_p}$
Unit Ramp t	$\dfrac{1}{K_v}$
Parabolic $\dfrac{1}{2}t^2$	$\dfrac{1}{K_a}$

$$K_p = \lim_{s \to 0} G(s)$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

Error constant and steady state error (1)

Forward path TF $$G(s)=\frac{K(s+z_1)\cdots(s+z_m)}{s^N(s+p_1)\cdots(s+p_Q)}$$

$$K_p=\lim_{s\to 0}G(s)=\lim_{s\to 0}\frac{1}{s^N}\lim_{s\to 0}\frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_Q)}$$

$$=\frac{Kz_1\cdots z_m}{p_1\cdots p_Q}\lim_{s\to 0}\frac{1}{s^N}=\begin{cases}\dfrac{Kz_1\cdots z_m}{p_1\cdots p_Q} & N=0\\ \infty & N\geq 1\end{cases}$$

For unit step input $$e_{ss}=\frac{1}{1+K_p}$$

Error constant and steady state error (2)

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{1}{s^{N-1}} \lim_{s \to 0} \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_Q)}$$

$$= \frac{Kz_1 \cdots z_m}{p_1 \cdots p_Q} \lim_{s \to 0} \frac{1}{s^{N-1}} = \begin{cases} 0 & N=0 \\ \dfrac{Kz_1 \cdots z_m}{p_1 \cdots p_Q} & N=1 \\ \infty & N>1 \end{cases}$$

For unit ramp input

$$e_{ss} = \frac{1}{K_v} = \begin{cases} \infty & N=0 \\ \dfrac{p_1 \cdots p_Q}{Kz_1 \cdots z_m} & N=1 \\ 0 & N>1 \end{cases}$$

Error constant and steady state error (3)

$$K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{1}{s^{N-2}} \lim_{s\to 0} \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_Q)}$$

$$= \frac{Kz_1\cdots z_m}{p_1\cdots p_Q} \lim_{s\to 0} \frac{1}{s^{N-2}} = \begin{cases} 0 & N=0,1 \\ \dfrac{Kz_1\cdots z_m}{p_1\cdots p_Q} & N=2 \\ \infty & N>2 \end{cases}$$

For unit parabolic input

$$e_{ss} = \frac{1}{K_a} = \begin{cases} \infty & N=0,1 \\ \dfrac{p_1\cdots p_Q}{Kz_1\cdots z_m} & N=2 \\ 0 & N>2 \end{cases}$$

System Type Number

$$G(s) = \frac{K(s+z_1)\cdots(s+z_m)}{s^N(s+p_1)\cdots(s+p_Q)} = \left(\frac{1}{s}\right)^N \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_Q)}$$

- The number of ***integration*** in the forward path (open-loop) TF is important in determining steady state errors!

N: ***system type number***

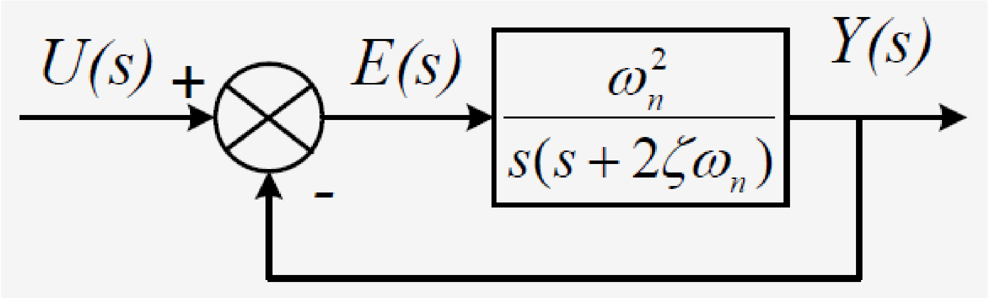
$$G(s) = \frac{K}{s+1}$$ Type 0 system

$$G(s) = \frac{K}{s(s+1)}$$ Type 1 system

$$G(s) = \frac{K}{s^2(s+1)}$$ Type 2 system

Steady State Errors of 2nd Order System

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$



- One integration in open-loop TF: *Type 1 system*!

Steady state errors:

input signal	steady state error
unit step	0
unit ramp	$2\zeta/\omega_n$
unit parabolic	∞

$$K_p = \lim_{s\to 0} G(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{\omega_n}{2\zeta}$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

Design Example

- A motor-driven optical tracking system has the unity negative feedback configuration with the forward TF as

$$G(s) = \frac{12K_o}{s(s+10)}$$

- The desired tracking performance is that the steady-state error is less than *0.01 rad* with an input signal that is changing at a *constant rate* of angular variation of *0.06 rad/s*.
  - Find the range for K_o.

$$u(t) = 0.06t \; rad / s \qquad e_{ss} < 0.01 \; rad$$

ramp input $\qquad |y(\infty) - u(\infty)| < 0.01 \; rad$

Design Example: Solution

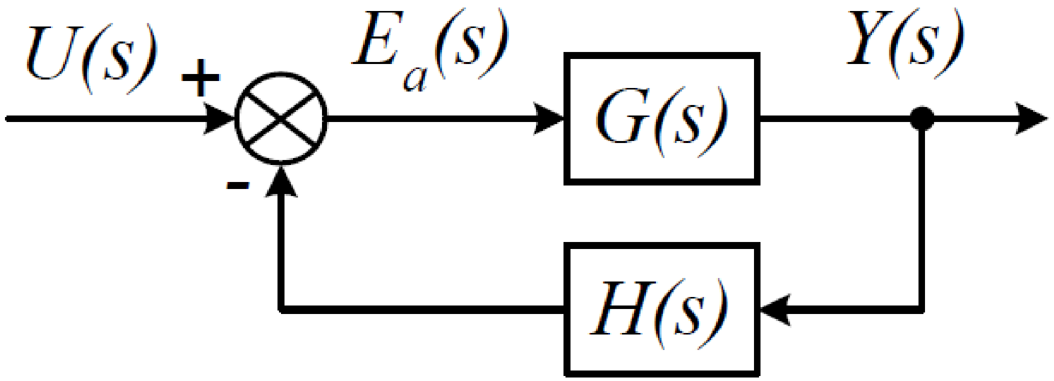
$$u(t) = 0.06t \ rad/s \qquad e_{ss} = 0.06\frac{1}{K_v} \qquad G(s) = \frac{12K_o}{s(s+10)}$$

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\frac{12K_o}{s(s+10)} = 1.2K_o$$

$$e_{ss} = \frac{0.06}{1.2K_o} < 0.01 \Rightarrow \frac{6}{1.2K_o} < 1$$

$$\Rightarrow \frac{5}{K_o} < 1 \Rightarrow K_o > 5$$

Steady state errors for nonunity feedback systems (1)



- Note: $E_a(s)$ is NOT error signal!

Error: $E(s) = U(s) - Y(s)$

- **Method**: Directly use the error definition and the final value theorem to find the steady state error.