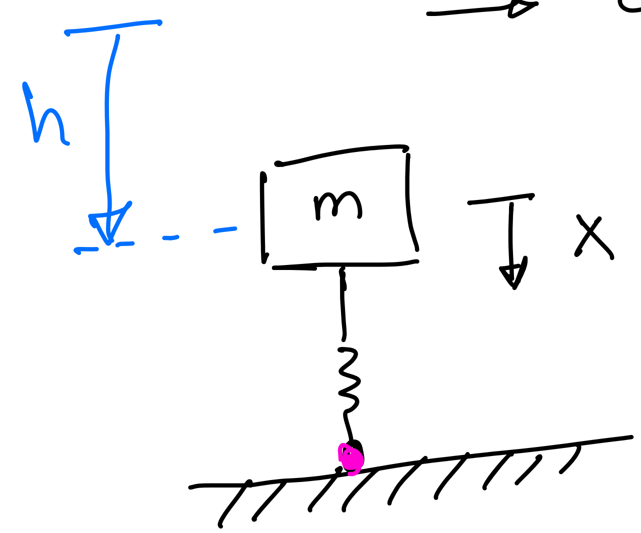
Example 4.3-2 (Drop test)

The question of how far a body can be dropped without causing damage.

Application → landing of airplanes.

→ cushioning of packages.



$v^2 - v_0^2 = 2gh$

Initial velocity = 0

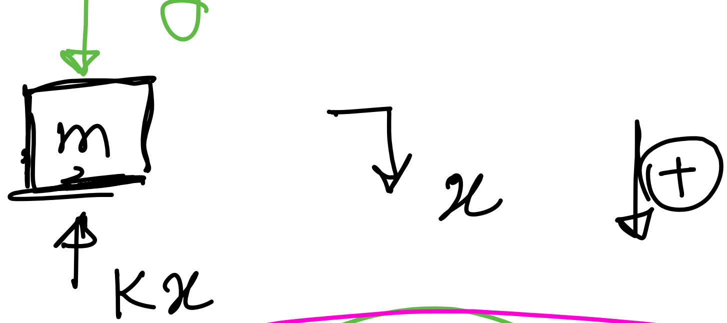
$\dot{x}(0) = V = \sqrt{2gh}$

x → is from the position of mass, m, at time $t = 0$

h → mass is dropped from height h

mg

F.B.D.

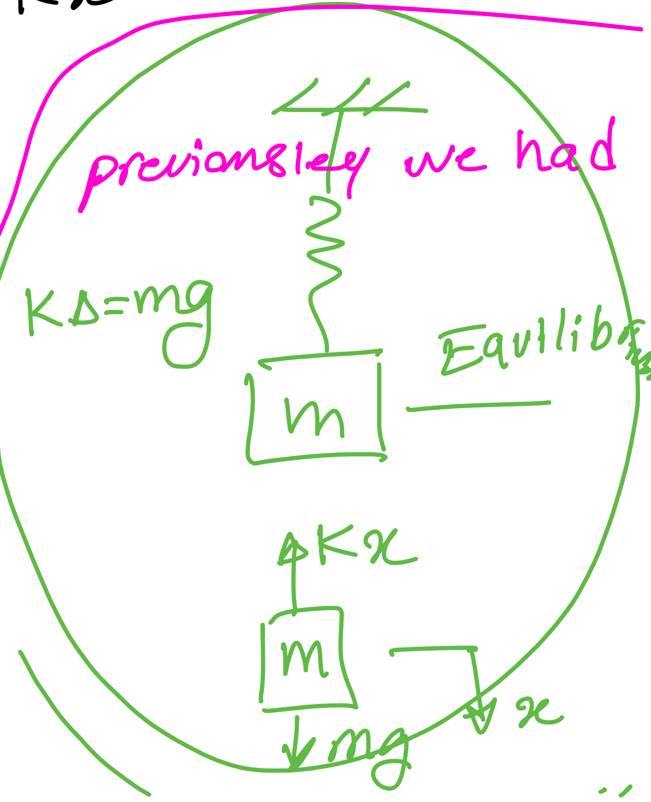


Eq. of motion:

$$\oplus\downarrow \sum F = m\ddot{x}$$

$$mg - Kx = m\ddot{x}$$

$$m\ddot{x} + Kx = mg$$



$$mg - K(x+\Delta) = m\ddot{x}$$

$$\cancel{mg} - Kx - \cancel{K\Delta} = m\ddot{x}$$

Taking Laplace

with initial conditions

$$\boxed{x(0) = 0}$$

$$\dot{x}(0) = \sqrt{2gh}$$

$$\mathcal{L}[\ddot{x}] = s^2X(s) - s\,\underbrace{\cancel{x(0)}}_{0} - \dot{x}(0)$$

Laplace

$$m[s^2X(s) - \dot{x}(0)] + KX(s) = \mathcal{L}[mg]$$

Laplace of (mg)

$$(m\,g)\mathcal{L}[1] = mg\left(\frac{1}{s}\right)$$

$$m\left[s^2X(s) - \sqrt{2gh}\right] + KX(s) = \frac{mg}{s}$$

Laplace Reminder

$$y(t) \xrightarrow{\mathcal{L}} \mathcal{L}(y(t)) = Y(s)$$

$$\mathcal{L}[\dot{y}(t)] = sY(s) - y(0)$$

$$\mathcal{L}[\ddot{y}(t)] = s^2Y(s) - y(0)s - \dot{y}(0)$$

$$X(s)\left[ms^2 + K\right] = m\sqrt{2gh} + \frac{mg}{s}$$

$$X(s) = \frac{m\sqrt{2gh}}{[ms^2+K]} + \frac{mg}{s[ms^2+K]}$$

Divide by m:

$$\boxed{\frac{K}{m} = \omega_n^2}$$

$$X(s) = \frac{\sqrt{2gh}}{s^2+\omega_n^2} + \frac{g}{s(s^2+\omega_n^2)}$$

We found X in Laplace domain?

Look up $\frac{1}{s^2+w_n^2}$ in Laplace table

constant

and find the time domain or

$\mathcal{L}^{-1}\left[\frac{1}{s^2+w_n^2}\right]$ Laplace inverse.

$$\mathcal{L}^{-1}[X(s)]=\mathcal{L}^{-1}\left[\frac{\sqrt{2gh}}{s^2+w_n^2}+\frac{g}{s(s^2+w_n^2)}\right]$$

$$x(t)=\mathcal{L}^{-1}\left[\frac{\sqrt{2gh}}{s^2+w_n^2}\right]+\mathcal{L}^{-1}\left[\frac{g}{s(s^2+w_n^2)}\right]$$

$$x(t)=\sqrt{2gh}\,\mathcal{L}^{-1}\left[\frac{1}{s^2+w_n^2}\right]+g\,\mathcal{L}^{-1}\left[\frac{1}{s(s^2+w_n^2)}\right]$$

From the Laplace table:

Laplace	Time
$\dfrac{a}{s^2+a^2}$	$\sin at$

Laplace	Time
$\dfrac{1}{s^2+a^2}$	$\dfrac{1}{a}\sin at$

Book

In our equation $a = w_n^2$

$$\frac{\sqrt{2gh}}{w_n}\mathcal{L}^{-1}\left[\frac{w_n}{s^2+w_n^2}\right] = \frac{\sqrt{2gh}}{w_n}\sin w_n t$$

$$\frac{1}{s(s^2+w_n^2)} = \left[\frac{A}{s} + \frac{Bs+C}{s^2+w_n^2}\right]$$

$$1 = A(s^2+w_n^2) + Bs^2 + Cs$$

$$1 = (A+B)s^2 + Cs + Aw_n^2$$

Campare both sides of the equatian and equate the equivalent terms by the order of S :

	Left hand side of the Eq (HSE)	Right HSE
S^2	0	$A+B$
S	0	C
	1	$A\omega_n^2$

$$\begin{cases} A+B=0 \\ C=0 \\ A\omega_n^2=1 \end{cases} \Rightarrow \begin{cases} A=\dfrac{1}{\omega_n^2} \\ B=-\dfrac{1}{\omega_n^2} \end{cases}$$

$$\left[\frac{A}{S}+\frac{BS+C}{S^2+\omega_n^2}\right]=\frac{1}{\omega_n^2 S}-\frac{S}{\omega_n^2\{S^2+\omega_n^2\}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{\omega_n^2 s} - \frac{s}{\omega_n^2\{s^2+\omega_n^2\}}\right]$$

$$= \frac{1}{\omega_n^2}\,\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$$

$$= \frac{1}{\omega_n^2}\,\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{\omega_n^2}\,\underbrace{\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega_n^2}\right]}_{\text{Laplace Table}}$$

$$= \frac{1}{\omega_n^2}(1) - \frac{1}{\omega_n^2}\left(\cos \omega_n t\right)$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{\sqrt{2gh}}{s^2+\omega_n^2}\right] + \mathcal{L}^{-1}\left[\frac{g}{s(s^2+\omega_n^2)}\right]$$

$$x(t) = \frac{\sqrt{2gh}}{\omega_n} \sin \omega_n t + \frac{g}{\omega_n^2}(1 - \cos \omega_n t)$$

$$x(t) = \left(\frac{\sqrt{2gh}}{\omega_n}\right) \sin \omega_n t - \frac{g}{\omega_n^2} \cos \omega_n t$$

$$+ \frac{g}{\omega_n^2}$$

$$\boxed{\begin{cases} \sin(a+b) = \sin a \sin b + \cos a \cos b \\ \sin(a-b) = -\sin a \sin b + \cos a \cos b \end{cases}}$$

$$x(t) = \frac{\sqrt{2gh}}{\omega_n} \sin \omega_n t + \frac{g}{\omega_n^2}(1 - \cos \omega_n t)$$

$$= \sqrt{\frac{2gh}{\omega_n^2} + \left(\frac{g}{\omega_n^2}\right)^2} \sin(\omega_n t - \phi)$$

$$+ \frac{g}{\omega_n^2}$$

$$\dot{x}(t) = \omega_n \sqrt{\frac{2gh}{\omega_n^2} + \left(\frac{g}{\omega_n^2}\right)^2} \cos(\omega_n t - \phi)$$

$$\ddot{x}(t) = -\omega_n^2 \sqrt{\frac{2gh}{\omega_n^2} + \left(\frac{g}{\omega_n^2}\right)^2} \sin(\omega_n t - \phi)$$

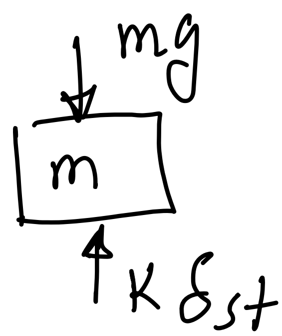
Max. displacement and acceleration occur at $\sin(\omega_n t - \phi) = 1$

Max acceleration:

$$\frac{\ddot{x}}{g} = -\sqrt{\frac{2h}{\delta_{st}} + 1}$$

δ_{st} = static equilibrium

static : $K\delta_{st} = mg$

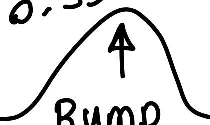


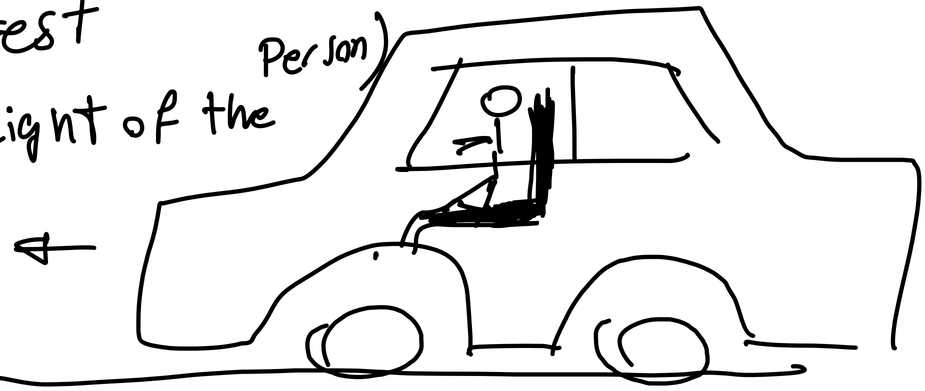
Example 4.3-3

Biomedical test

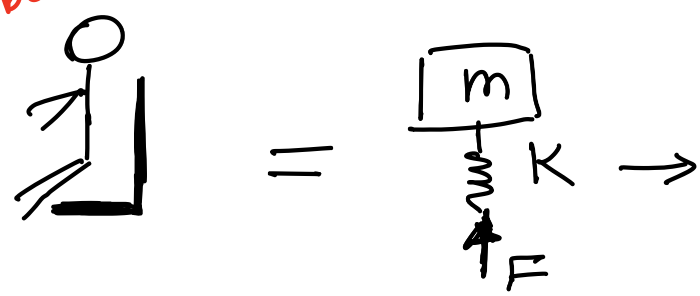
$mg = 160\ \text{lb}$ (weight of the Person)

$\delta_{st} = \dfrac{mg}{K} = 0.35\ \text{in}$





No seat belt (Do not do this at home)!



$K =$ the spinal stiffness

$= 81{,}000\ \dfrac{N}{m}$

$= 458\ \dfrac{\text{lb}}{\text{in}}$

The person thrown upward when the vehicle hit the obstacle/bump and drops to the seat with $h = 3$ in (free fall)

$$\frac{\ddot{x}}{g} = -\sqrt{\frac{2h}{\delta_{st}} + 1} = -\sqrt{\frac{2 \times 3}{0.35} + 1} = -4.26$$