Instrumentation and Controls
ETM 3301

Lecture 15

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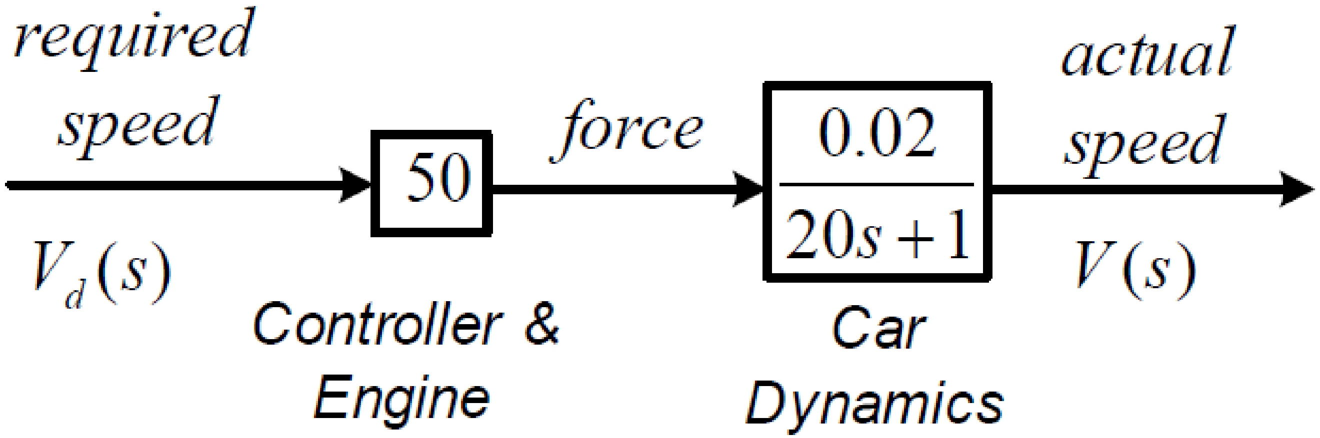
Chapter 6: Steady State Errors of Control Systems

- *Steady state errors* and calculation using *final value theorem*.
- Error constants for unity negative feedback systems and steady state errors.
- System type number for unity negative feedback systems.
- Steady state errors for systems with external disturbances .

Car Speed Control Problem



Desired speed $v_d(t) = 30m/s$ $V_d(s) = \frac{30}{s}$

Find actual speed $v(t)$

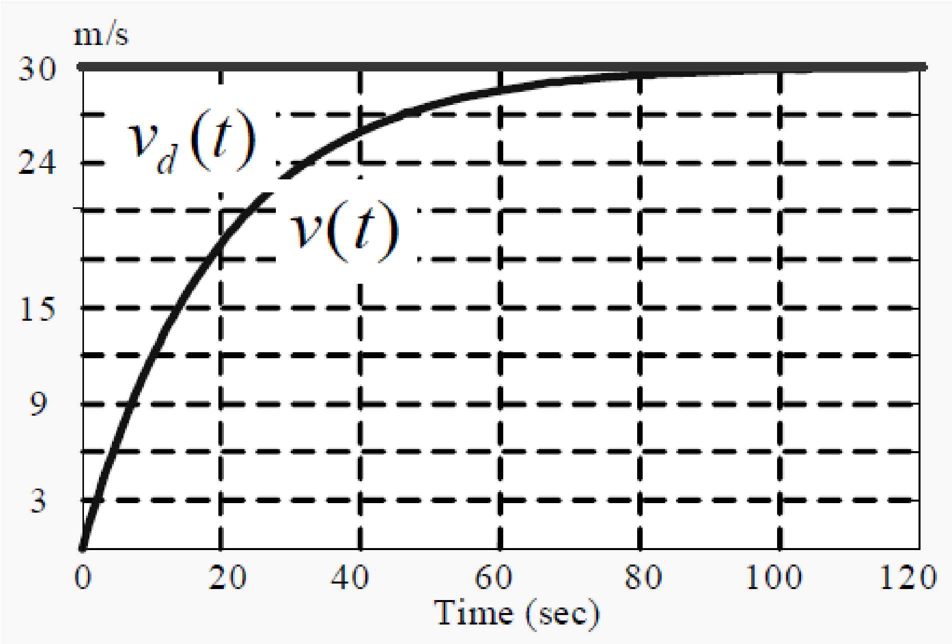
$$V(s) = 50\frac{0.02}{20s+1}V_d(s) = 30\frac{1}{s(20s+1)}$$

$$= 30\frac{1+20s-20s}{s(20s+1)} = 30\left(\frac{1}{s} - \frac{20}{20s+1}\right)$$

Car Speed Control Problem

$$V(s) = 30\left(\frac{1}{s} - \frac{20}{20s+1}\right) = 30\left(\frac{1}{s} - \frac{1}{s+0.05}\right)$$

$$v(t) = L^{-1}(V(s)) = 30\left(1 - e^{-0.05t}\right)$$



Speed control (tracking) error:

$$e(t) = v_d(t) - v(t) = 30e^{-0.05t}$$

Car Speed Control Problem

Steady state speed control (tracking) error:

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} 30e^{-0.05t} = 0$$

Alternative method to find steady state error:

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)}$$

Final value theorem

$$e(t) = v_d(t) - v(t)$$

$$E(s) = V_d(s) - V(s)$$

$$= \frac{30}{s} - \frac{30}{s(20s+1)}$$

Car Speed Control Problem

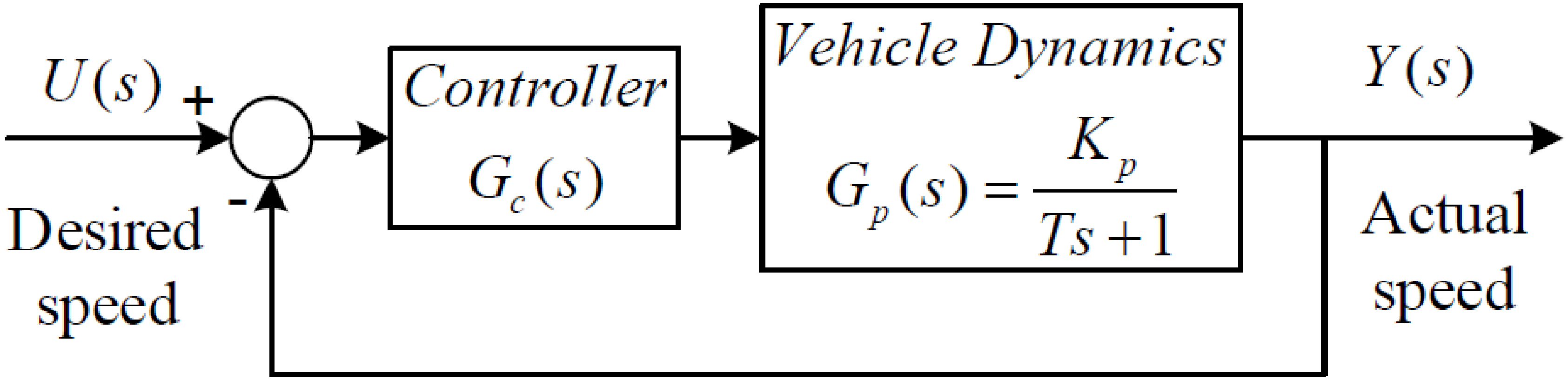
Steady state speed control (tracking) error:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\left(\frac{30}{s} - \frac{30}{s(20s+1)}\right)$$

$$= \lim_{s\to 0}\left(30 - \frac{30}{20s+1}\right) = 30 - \frac{30}{0+1} = 0$$

Steady State Errors

Example: Car Speed control example (closed-loop)



- Closed-loop system transfer function

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{K}{Ts + (K+1)}$$

$$G_c(s) = K_c \qquad K_c K_p = K$$

Steady State Errors

- If the required speed is 30m/s ($u(t)=30$), the system response is:

$$Y(s)=T(s)U(s)=\frac{K}{Ts+(K+1)}\frac{30}{s}$$

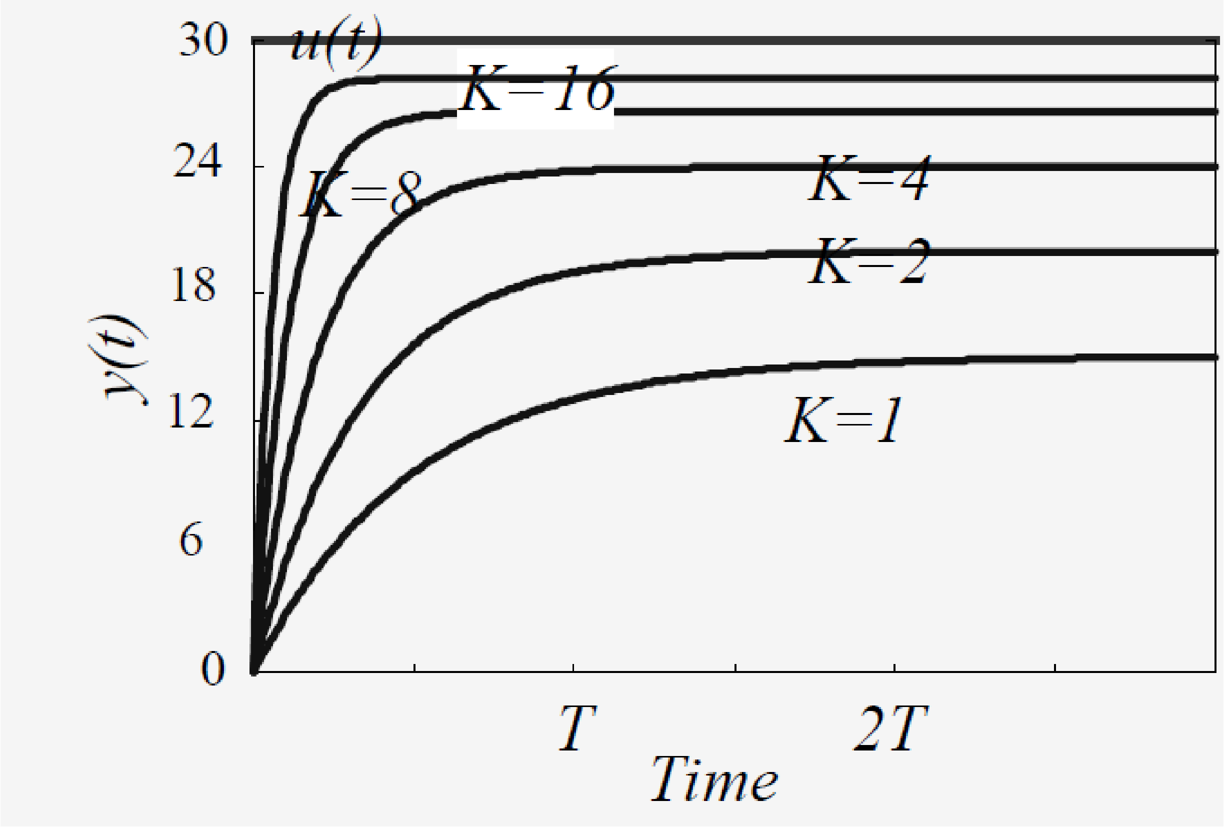
- Error (difference between input and output) is:

$$E(s)=U(s)-Y(s)=\frac{30}{s}-\frac{K}{Ts+(K+1)}\frac{30}{s}$$

- Steady state error (final value of the error):

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=30\frac{1}{K+1}$$

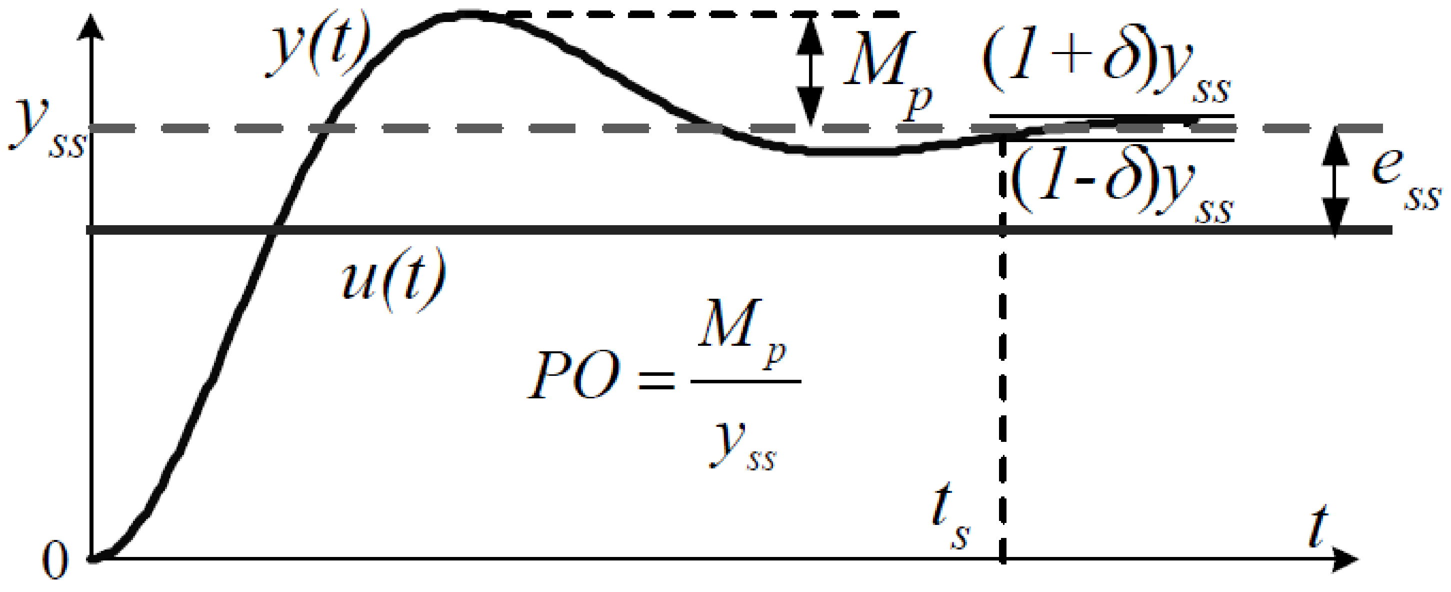
Steady State Errors



- Note: as *K* increases,
  - the response gets faster
  - the steady state error becomes smaller, but cannot be zero for finite *K*.

Control System Performance Indices

- Transient performance:
  - Percentage overshoot (PO) — *Smoothness*
  - Settling time (t_s) — *Speed*
- Steady state error (e_{ss}) — *Accuracy*



Steady State Error Definition

- Steady state error is the difference between input and output as $t \to \infty$.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} \big(u(t) - y(t) \big)$$

  - The steady state error is normally evaluated using standard test signals.

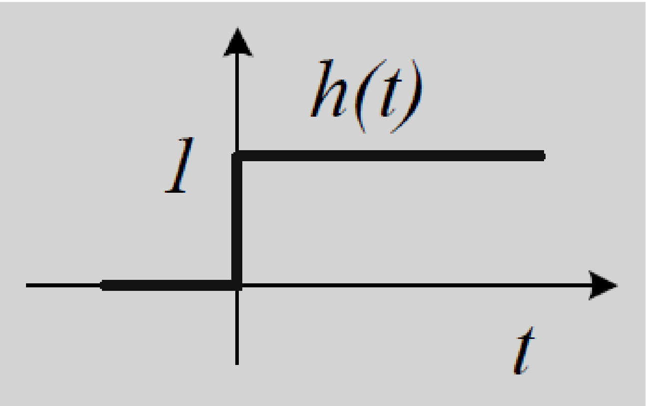
- The steady state error can be find using the Final Value Theorem

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

  - When $E(s)$ is known.

Standard Test Signals (1)

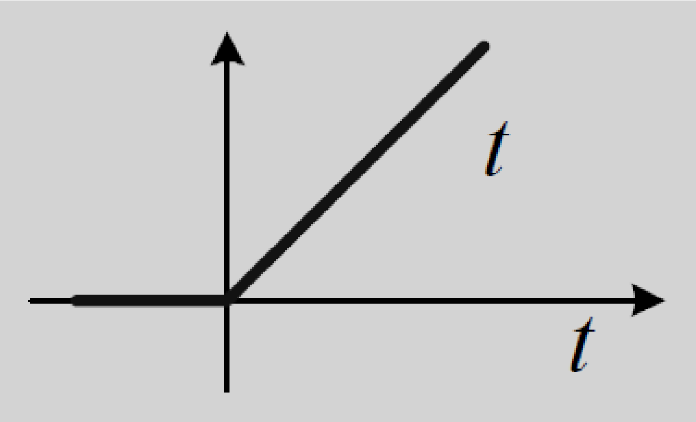
- **Unit Step input:**



$$u(t) = h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

- **Unit Ramp input:**

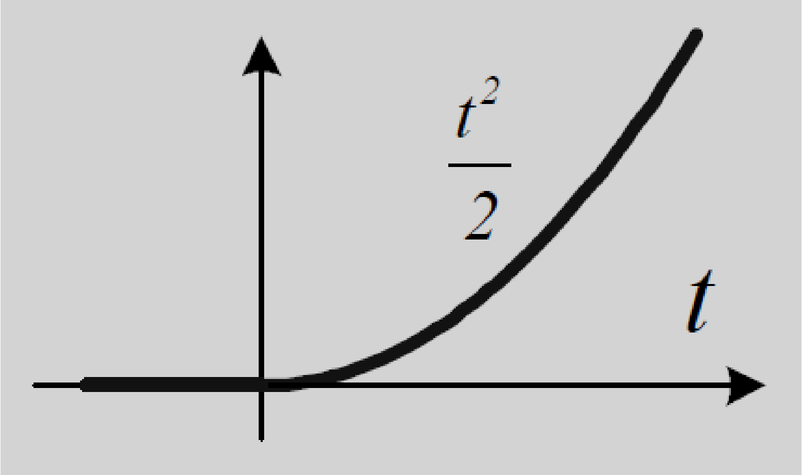


$$u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s^2}$$

Standard Test Signals (2)

- **Unit Parabolic input :**



$$u(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s^3}$$

Speed Control Example

Change Controller Transfer Function

- Closed-loop system transfer function

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{K(1+\alpha s)}{Ts^2 + (1+\alpha K)s + K}$$

$$G_c(s) = \frac{K_c(1+\alpha s)}{s} \qquad K_c K_p = K$$

- Apply a unity step ($u(t)=1$) to the system

$$U(s) = \frac{1}{s} \quad Y(s) = T(s)U(s) = \frac{K(1+\alpha s)}{Ts^2 + (1+\alpha K)s + K}\frac{1}{s}$$

Speed Control Example

- Error (difference between input and output) is:

$$e(t) = u(t) - y(t)$$

$$E(s) = U(s) - Y(s) = \frac{1}{s} - \frac{K(1+\alpha s)}{Ts^2 + (1+\alpha K)s + K}\frac{1}{s}$$

$$= \frac{1}{s}\left[1 - \frac{K(1+\alpha s)}{Ts^2 + (1+\alpha K)s + K}\right]$$

$$= \frac{1}{s}\frac{Ts^2 + (1+\alpha K)s + K - K(1+\alpha s)}{Ts^2 + (1+\alpha K)s + K}$$

$$= \frac{1}{s}\frac{Ts^2 + s}{Ts^2 + (1+\alpha K)s + K}$$

Speed Control Example

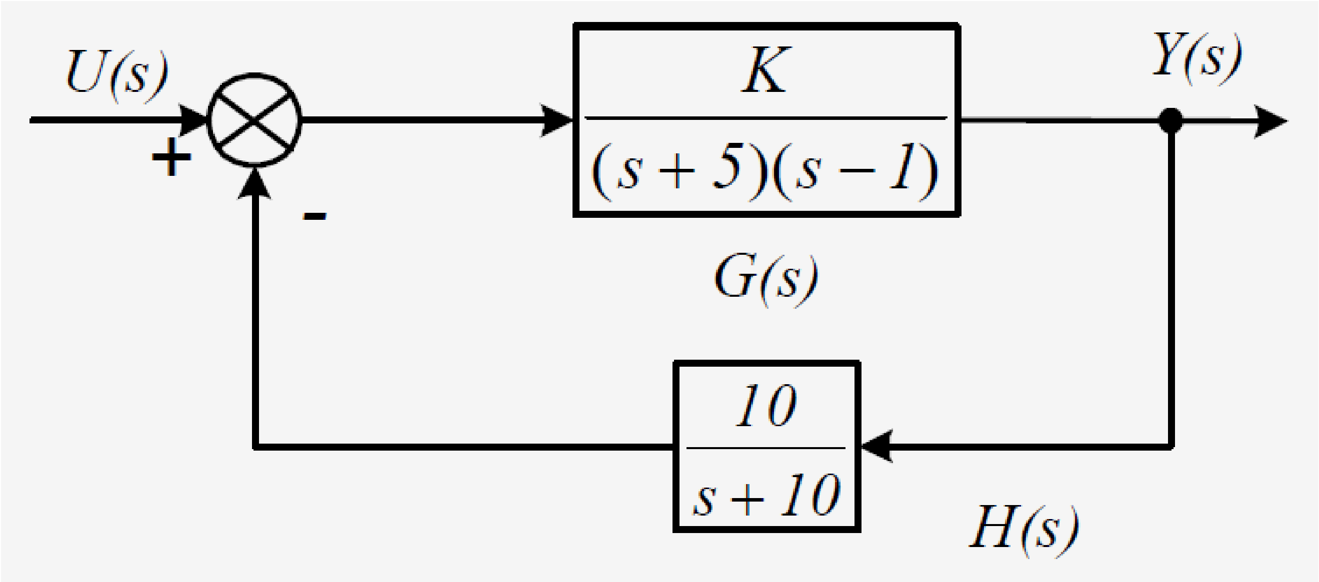
$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s\left(\frac{1}{s}\frac{Ts^2+s}{Ts^2+(1+\alpha K)s+K}\right)$$

$$= \lim_{s\to 0}\left(\frac{Ts^2+s}{Ts^2+(1+\alpha K)s+K}\right) = \frac{T0+0}{T0+0+(K+1)} = 0$$

Steady State Error Example

- Find the steady state error for a unit step input and discuss the way of reducing it.



Closed loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K(s+10)}{s^3 + 14s^2 + 35s + (10K - 50)}$$

Steady State Error Example

- Input: $u(t) = 1, \quad U(s) = \frac{1}{s}$
- Output: $Y(s) = T(s)U(s)$
- Error:

$$E(s) = U(s) - Y(s) = [1 - T(s)]U(s)$$

$$= \left[1 - \frac{K(s+10)}{s^3 + 14s^2 + 35s + (10K - 50)} \right] \frac{1}{s}$$

$$= \frac{1}{s} \times \frac{s^3 + 14s^2 + 35s + (10K - 50) - K(s+10)}{s^3 + 14s^2 + 35s + (10K - 50)}$$

Steady State Error Example

- Steady State Error:

$$e_{ss} = e(\infty) = \lim_{s \to 0} sE(s) =$$

$$= \lim_{s \to 0} s \times \frac{1}{s} \times \frac{s^3 + 14s^2 + 35s + (10K - 50) - K(s+10)}{s^3 + 14s^2 + 35s + (10K - 50)}$$

$$= \frac{0 + 0 + 0 + (10K - 50) - K(s+10)}{0 + 0 + 0 + (10K - 50)}$$

$$= \frac{-50}{10K - 50} = \frac{-5}{K - 5}$$

- The steady state error can be reduced by increasing K, but it must satisfies stability condition

$$5 < K < 54$$