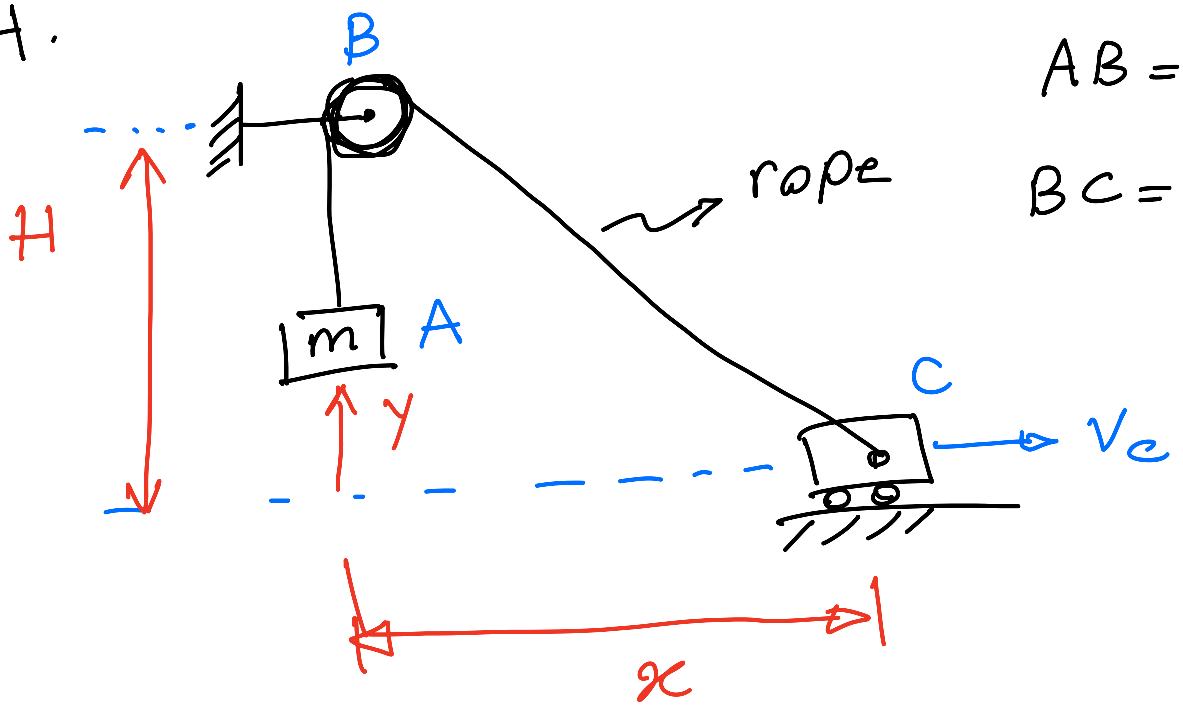
Example

Find the velocity of the mass m_A as a function of x, V_c and H.



$$AB = H - y$$

$$BC = \sqrt{x^2 + H^2}$$

The total length of the rope is constant $= L$

$$AB + BC = L$$

$$(H - y) + \sqrt{x^2 + H^2} = L$$

Take the derivative of Both sides:

$$\frac{d}{dt}(H-y) + \frac{d}{dt}\left(\sqrt{x^2+H^2}\right) = \frac{d}{dt}L$$

$u = x^2$

$x^2 + H^2 = g$

$f = f(g(u(x)))$

$$\cancelto{0}{\frac{d}{dt}H} - \frac{d}{dt}y + \frac{d}{dt}\left(f(g(u(x)))\right) = 0 \quad (*)$$

$$\sqrt{x^2+H^2} \quad \begin{cases} x^2 \to u & \to \quad du/dx = 2x \\ g \to u + H^2 & \to \quad dg/du = 1 \\ f \to \sqrt{g} & \to \quad \dfrac{df}{dg} = \dfrac{1}{2} g^{-1/2} \end{cases}$$

$$\frac{d}{dt}\sqrt{x^2+H^2} = \frac{d}{dt} f(g(u(x)))$$

$$= \frac{df}{dg}\frac{dg}{du}\frac{du}{dx}\frac{dx}{dt}$$

$$= \left(-\tfrac{1}{2} g^{-1/2}\right)(1)\,(2x)(\dot{x})$$

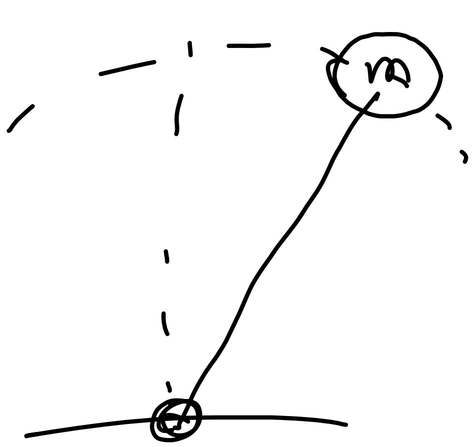
$g = u + H^2 \qquad u = x^2 \qquad \dot{x} = V_c$ velocity

$$= \frac{1}{2}\left(x^2+H^2\right)^{-1/2}(2x)(V_c)$$

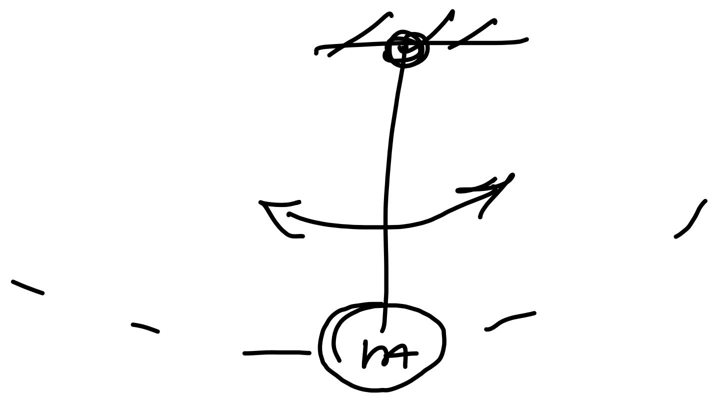
$$* \longrightarrow \frac{dy}{dt} = \frac{x V_c}{\sqrt{x^2+H^2}}$$

Example

Inverted pendulum.



unstable



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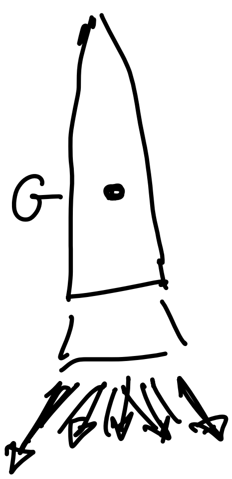

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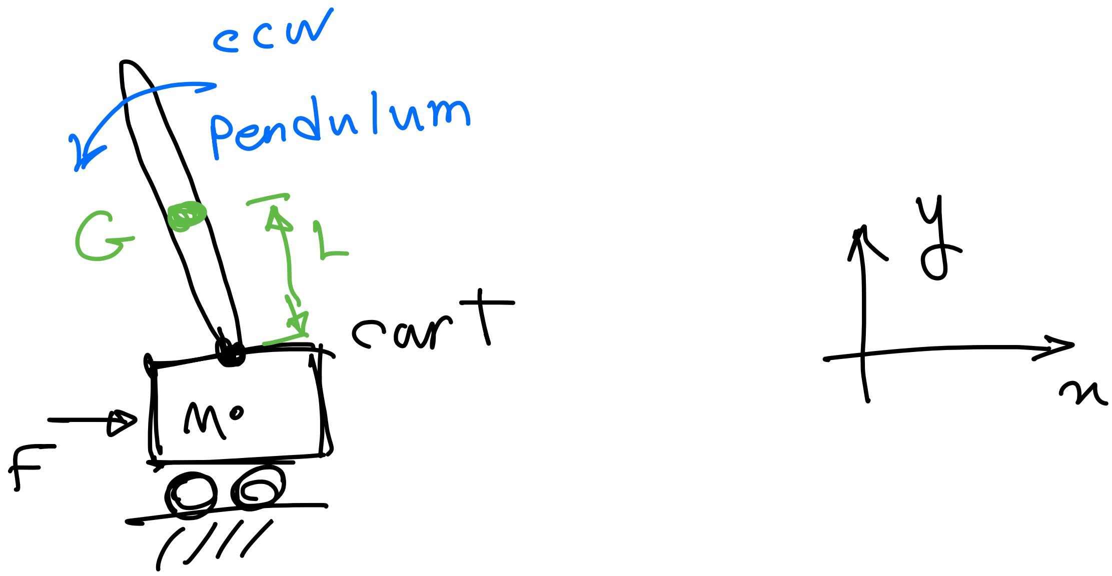
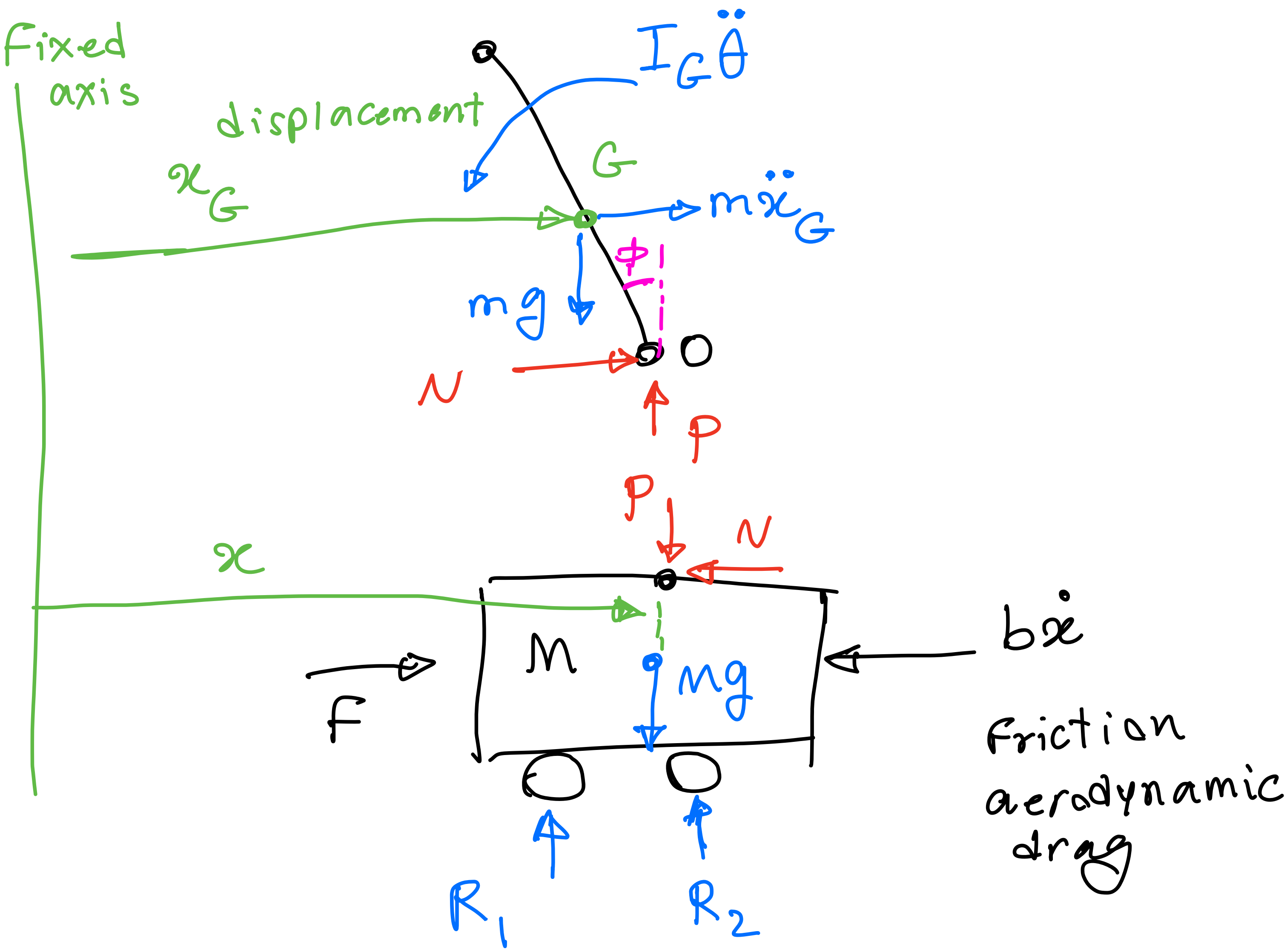


Free-body-diagram (F.B.D.)

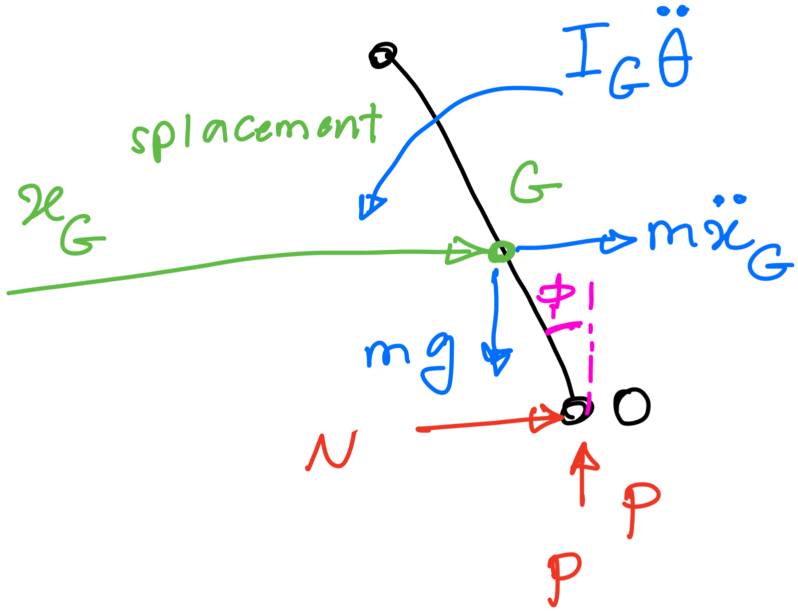


The cart:

$$\xrightarrow{+} \sum F_x = M\ddot{x}$$

$$F - b\dot{x} - N = M\ddot{x} \qquad (1)$$

The pendulum:



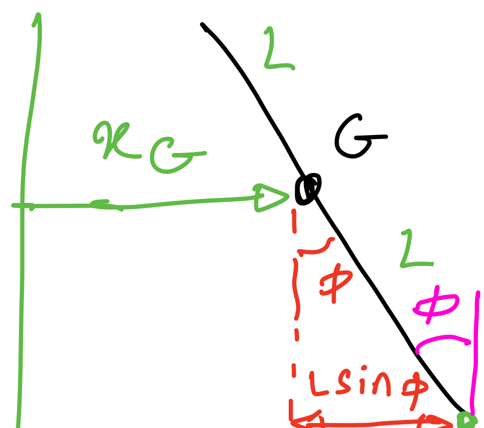
Translation:

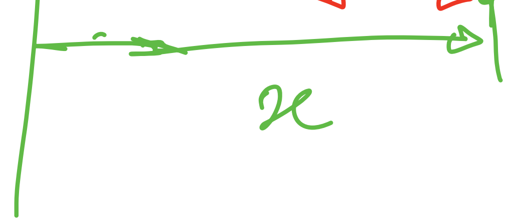
from FBD $\xrightarrow{+} \sum F_x = m\ddot{x}_G$

$$N = m\ddot{x}_G \qquad (2)$$

x_G in terms of x:

$$x_G = x - L\sin\phi$$





$$\dot{x}_G = \dot{x} - L \frac{d}{dt}\sin\phi$$

$$\dot{x}_G = \dot{x} - L \frac{d}{d\phi}\sin\phi \frac{d\phi}{dt}$$

$$\dot{x}_G = \dot{x} - L\cos\phi\,\dot{\phi}$$

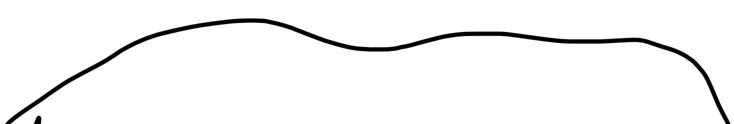
$$\ddot{x}_G = \ddot{x} - L \frac{d}{dt}(\cos\phi\,\dot{\phi})$$

$$\ddot{x}_G = \ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right] \quad (3)$$

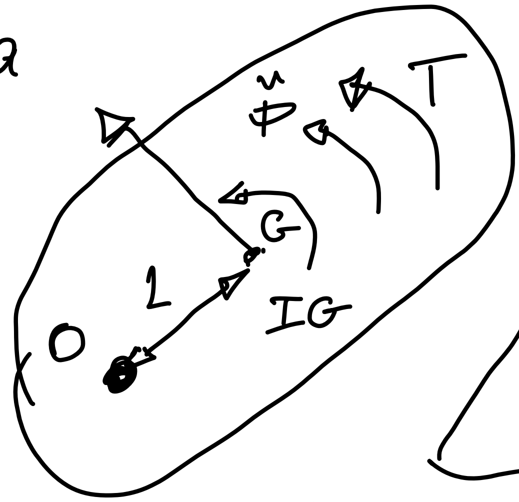
(2) & (3) $\Rightarrow$

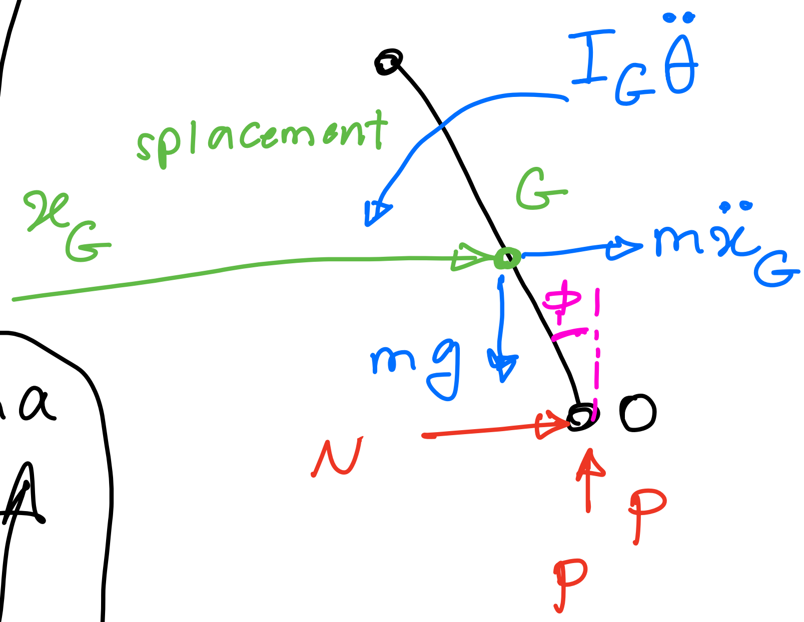
$$N = m\ddot{x}_G$$

$$N = m\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\}$$



$m\,a$





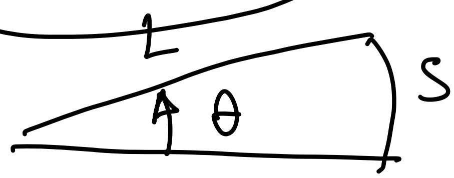
$$\Sigma M_O = I_G\ddot{\Phi} + L\,m\,a$$

$$\Sigma M_G = I_G\ddot{\Phi}$$

$$\Sigma M_O = I_O\ddot{\Phi}$$

$$a = L\ddot{\Phi}$$

$$\circlearrowleft(+)\ \Sigma M_O = I_G\ddot{\Phi}$$



$$s = L\theta$$

$$\dot{s} = L\dot{\theta}$$

$$\ddot{s} = L\ddot{\theta}$$

$$I_G\ddot{\Phi} + Lm(L\ddot{\Phi})$$

$$\underbrace{(I_G + mL^2)}_{\text{parallelaxis theorm}}\ddot{\Phi}$$

$$\circlearrowleft(+)\ \Sigma M_O = I_G\ddot{\Phi} - m\ddot{x}_G\cos\Phi\, L$$

small rotations:

If ϕ is small

$$\sin\phi = \phi$$
$$\cos\phi = 1$$
$$\dot{\phi}^2 = 0$$

It is desired that ϕ to remain small in practice so that the Pendulum remains vertical and stable

These will simplify the equations