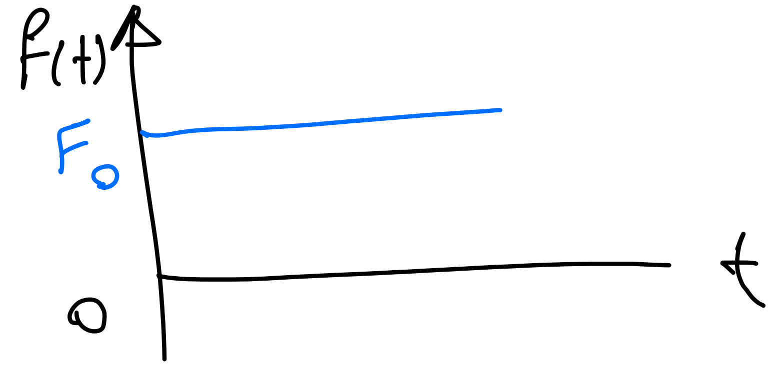
Example 4.2-1

Determine the response of a single-DOF system to the step excitation shown below



consider the undamped system

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

substitute in Eq. (4.2-1)

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n (t - \xi)\, d\xi$$

$$= \frac{F_0}{K} (1 - \cos \omega_n t)$$

For damped system, by repeating the procedure we have

$$h(t) = \frac{e^{-\xi \omega_n t}}{m\omega_n \sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\, \omega_n$$

for the differential equation

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = \frac{f}{m}$$

solution:

$$x(t) = X e^{-\xi \omega_n t} \sin\left(\sqrt{1-\xi^2}\, \omega_n t - \phi\right) + \frac{F_0}{m\omega_n^2}$$

Section 4.3 Laplace transform formulation. for solving differential equations. particularly finding the transfer function of a system (TF)

$$TF = \frac{\text{output}}{\text{Input}}$$

Example (4.3-1 Book)

The equation of motion of mass-spring-damper with force excitation of $F(t)$ is

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$

Taking the laplace transfrom:

from the Appendix:

$$f''(t) \xrightarrow[\text{Transform}]{\text{laplace}} s^2 F(s) \underbrace{- sf(0) - f'(0)}_{}$$

we usally don't have these initial condition terms.

$f'(0) = 0 \quad F(0) = 0$

$$f'(t) \xrightarrow{\text{Laplace}} sF(s) - \underline{f(0)}$$

$f(0) \to 0$

Force $\to$ Input

$x \to$ output

Respanse of the mass-spring-damper

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$

Laplace transform:

$$m\left(s^2 \bar{x}(s) - x(0)s - \dot{x}(0)\right)$$

$$+ c\left[s\bar{x}(s) - x(0)\right] + K\bar{x}(s)$$

$$= \bar{F}(s)$$

salving :

with initial conditions equal to zero we have:

$$m s^2 \bar{x}(s) + cs\bar{x}(s) + K\bar{x}(s) = \bar{F}(s)$$

$$\bar{x}(s) = \frac{\bar{F}(s)}{ms^2 + cs + K}$$

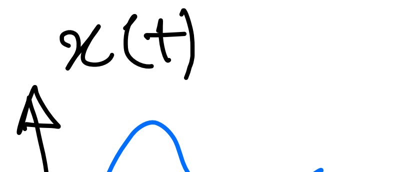
Transfer function $\left(\frac{\text{output}}{\text{Input}}\right)$

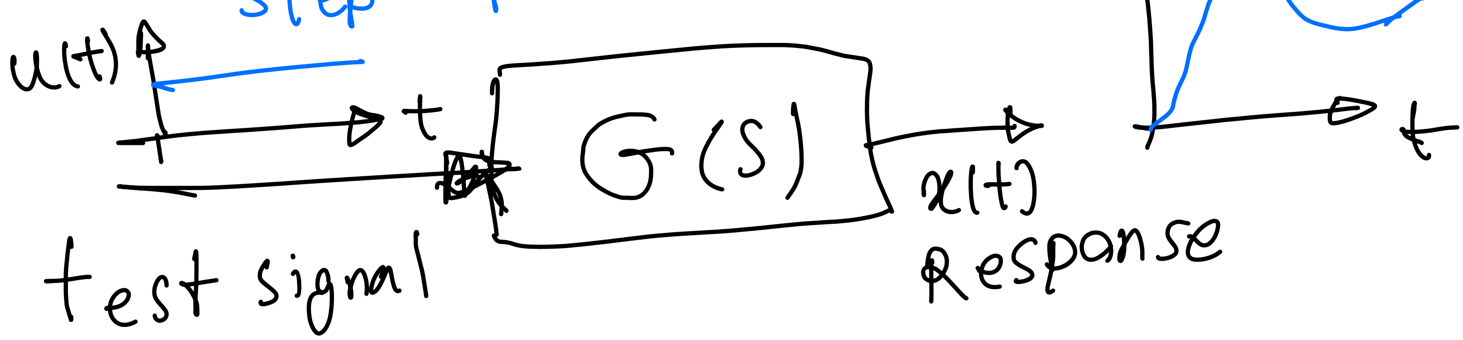
$$\frac{\bar{x}(s)}{\bar{F}(s)} = \frac{1}{ms^2 + cs + K}$$

Not using the Book for the rest of the Laplace transform

Time Responses

step input





the system response is normally evaluated using standard test input signals

standard test signals:

step input

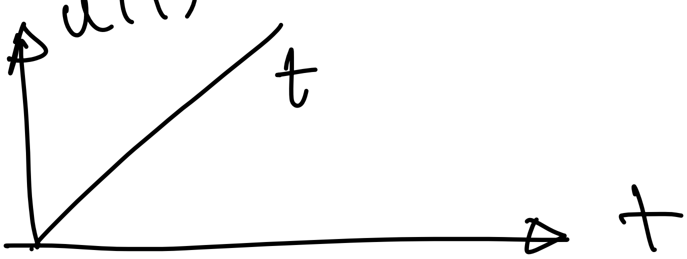
$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[h(t)] = \frac{1}{s}$$

Laplace

$$\mathcal{L}[1] = \frac{1}{s}$$

Ramp



$$u(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s^2}$$

$$u(t) = t \longrightarrow \text{Laplace} \longrightarrow \frac{1}{s^2}$$

mass - spring - damper system

example

$$\xrightarrow{\text{Input}} \boxed{G(s)} \xrightarrow{\text{output}}$$

$$G(s) = \frac{\text{output}}{\text{Input}} = \frac{X(s)}{U(s)}$$

$$M\ddot{x} + B\dot{x} + Kx = u(t)$$

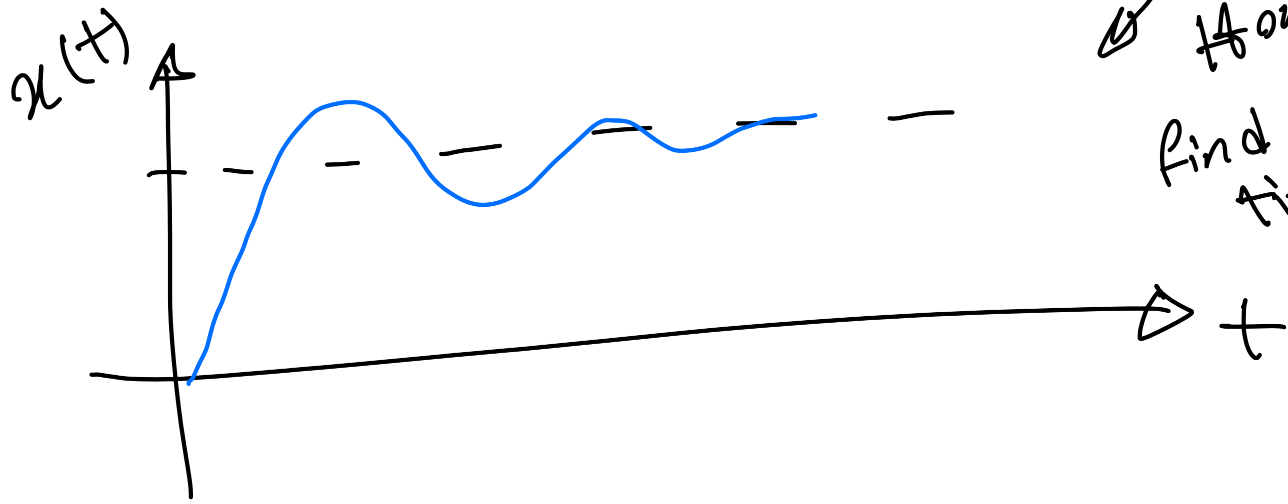
$$G(s) = \frac{1}{Ms^2 + Bs + K}$$

$M = 800$ kg $K = 400$ N/m

$B = 300$ Ns/m

$$G(S) = \frac{0.00125}{s^2 + 0.0375s + 0.5}$$

How to find the time Response



Example

Find the transfer function of $G(s)$ in time damain.

$$G(S) = \frac{Y(S)}{U(S)} = \frac{6}{s^2 + 5s + 6}$$

Input $u(t) = 1 \xrightarrow[\text{Laplace}]{} U(S) = \frac{1}{s}$

$$Y(s) = G(s)\,U(s)$$

$$Y(s) = \frac{6}{(s^2+5s+6)}\frac{1}{s}$$

$$Y(s) = \frac{6}{s(s^2+5s+6)}$$

partial fraction

$$Y(s) = \frac{6}{s(s+2)(s+3)} = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

$$Y(t) = 1 - 3e^{-2t} + 2e^{-3t}$$

Example

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4}{s^2+2s+1}$$ Transfer Funtion

$$u(t) = 1 \xrightarrow[\text{laplace}]{} U(s) = \frac{1}{s}$$ Input

output

$$Y(s) = \frac{4}{s(s^2+2s+1)} = \frac{4}{s(s+1)^2}$$

partial fraction $\rightarrow$

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$A, B, C = ?$

$A = 4$, $B = -4$, $C = -4$

$$Y(s) = \frac{4}{s} - \frac{4}{s+1} - \frac{4}{(s+1)^2}$$

using Laplace table

$$y(t) = 4(1 - e^{-t} - te^{-t})$$

$$\mathcal{L}^{-1}\, Y(s) \xrightarrow[\text{table}]{\text{using laplace}} y(t)$$

Laplace inverse

Response in time domain

second order transfer function

$$G(s) = \frac{1}{ms^2 + cs + K} = \frac{w_n^2}{s^2 + 2\underbrace{\xi}_{\text{Damping ratio}} w_n s + w_n^2}$$

$$U(s) \longrightarrow \boxed{\frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}} \longrightarrow Y(s)$$

$$s^2 + 2\xi w_n s + w_n^2 = 0$$

$$s^2 + 2\xi w_n + (\xi w_n)^2 - (\xi w_n)^2 + w_n^2 = 0$$

$$\Rightarrow (s + \xi w_n)^2 - (\xi^2 - 1) w_n^2 = 0$$

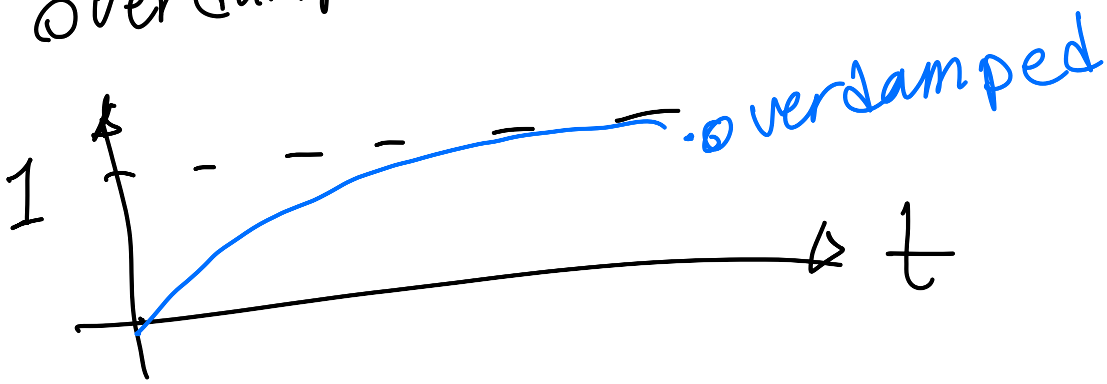
$$\Rightarrow (s + \xi w_n)^2 = (\xi^2 - 1) w_n^2$$

Case 1: $\xi > 1$

$$s_{1,2} = -\xi w_n \pm \sqrt{\xi^2 - 1}\, w_n$$

$\sqrt{\xi^2 - 1} > 1$

overdamped



case $\xi < 1$

$$s_{1,2} = -\xi w_n \pm j\sqrt{1 - \xi^2}\, w_n$$

under damped responce