Instrumentation and Controls

ETM 3301

Lecture 13

Instructor

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Chapter 5: Stability

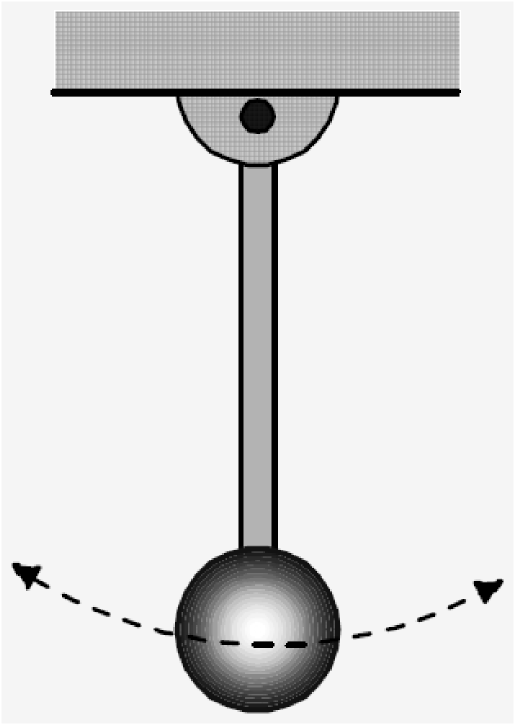
- What is a stable system?
- Relationship between time response shapes and pole locations.
- Routh-Hurwitz Stability Criterion.

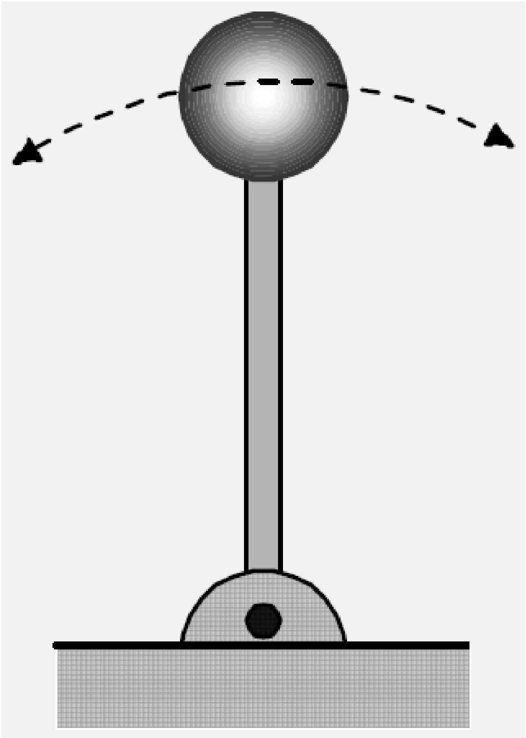
Stable

Cambridge Dictionary: "*firmly fixed or not likely to move or change*"

Oxford Dictionary: "*…(of an object or structure) not likely to give way or overturn; firmly fixed…*"

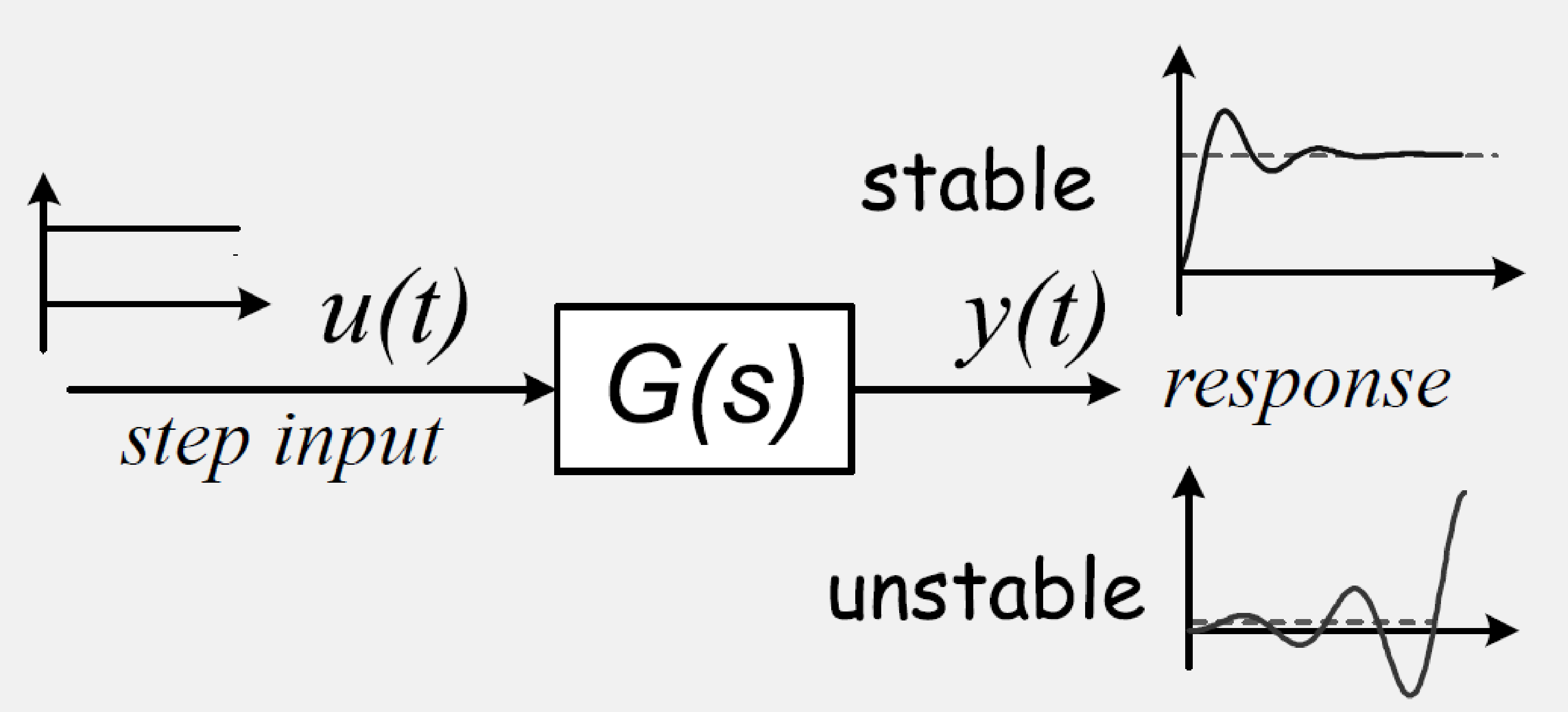
Stable and Unstable Systems





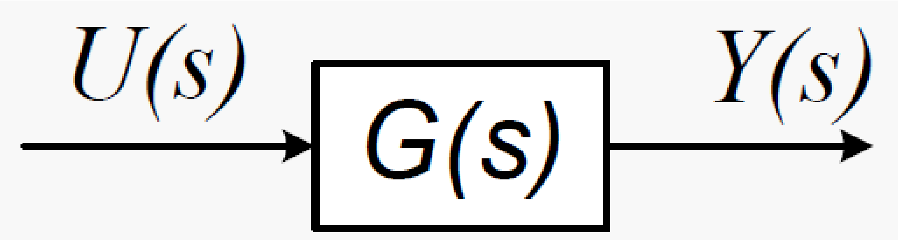
- Stable system:
  - the system can return to original state after a small disturbance.

Stable and Unstable System Step Responses



- Stable system:
  - the response gradually settles down to a required value.

Transfer Function and Poles



- The transfer function can be written as:

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- Poles:

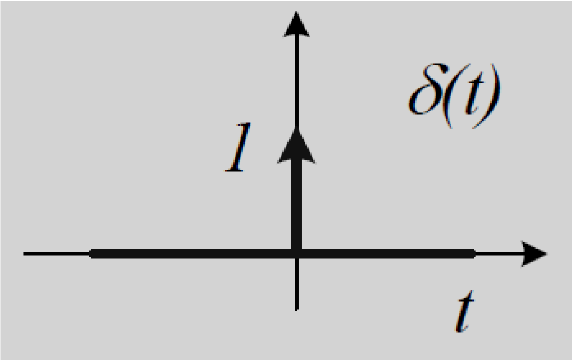
$$D(s) = (s+p_1)(s+p_2)\cdots(s+p_n) = 0$$

$$\Rightarrow \quad poles: \quad -p_1, -p_2, \cdots, -p_n$$

$$or \quad poles: \quad -p_i \ (i = 1, \cdots, n)$$

Impulse Response and Stability

- Disturbance can be represented by an impulse function.



$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\mathcal{L}[\delta(t)] = 1$$

- The Laplace transform of the output is

$$Y(s) = G(s)U(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Impulse Response and Stability

- Partial fraction expansion:
  - Assume that the system has distinct poles (real or complex conjugate). i.e. $p_i \neq p_j$ (when $i \neq j$)

$$Y(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \cdots + \frac{A_n}{s+p_n}$$

- Partial fraction expansion coefficients:

$$A_i = (s+p_i)\cdot Y(s)\Big|_{s=-p_i} \qquad i = 1,\cdots,n \qquad \text{cover-up rule}$$

Impulse Response and Stability

- Output time response:

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left(Y(s)\right) \\
&= \mathcal{L}^{-1}\left(\frac{A_1}{s+p_1}+\frac{A_2}{s+p_2}+\cdots+\frac{A_n}{s+p_n}\right) \\
&= A_1e^{-p_1t}+A_2e^{-p_2t}+\cdots+A_ne^{-p_nt}
\end{aligned}
$$

- Stable system: the system can return to original state after a disturbance

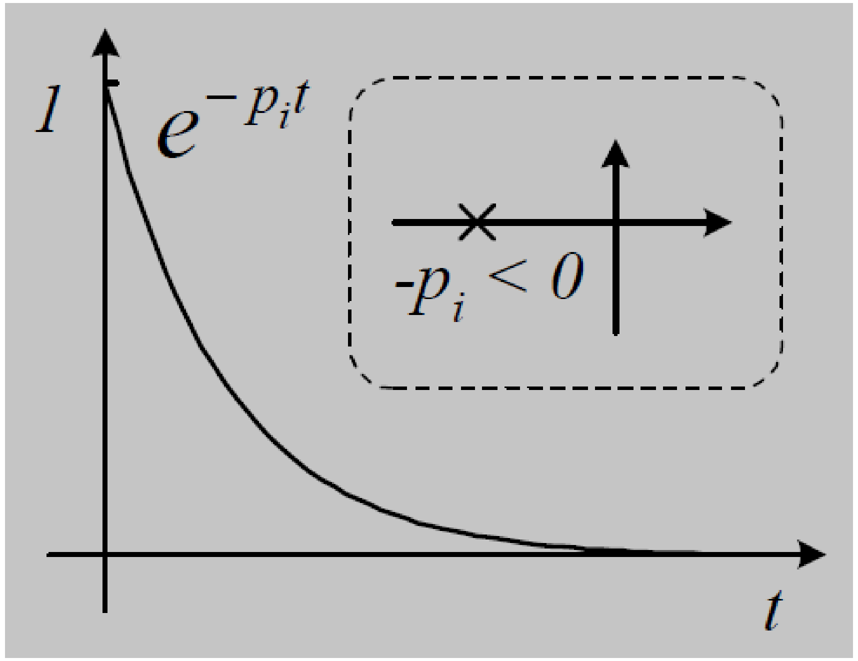
$$
stable \Leftrightarrow y(t)\to 0 \quad \Leftrightarrow \quad e^{-p_it}\to 0\ (i=1\cdots n)
$$

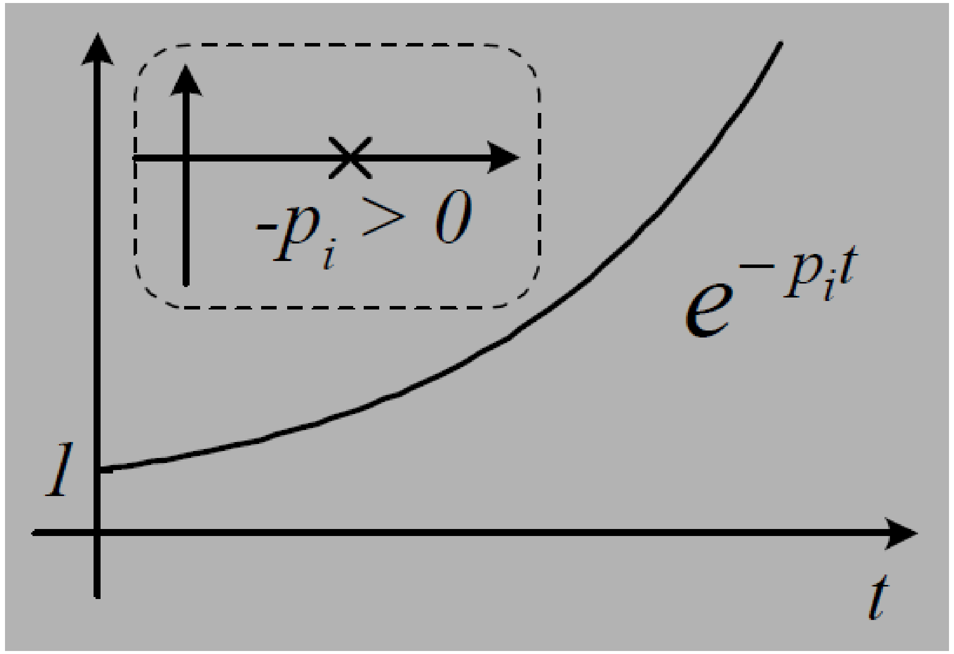
Modes, Transient Response and Pole Locations

$e^{-p_i t}$

- Defined as a ***mode*** of the transfer function $G(s)$

- Real poles ($-p_i$ is real, e.g. -2, 3)



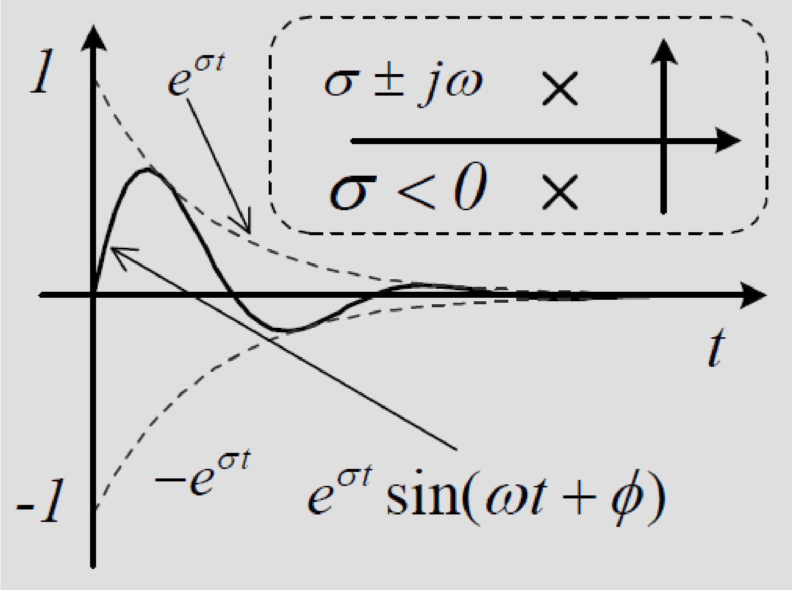


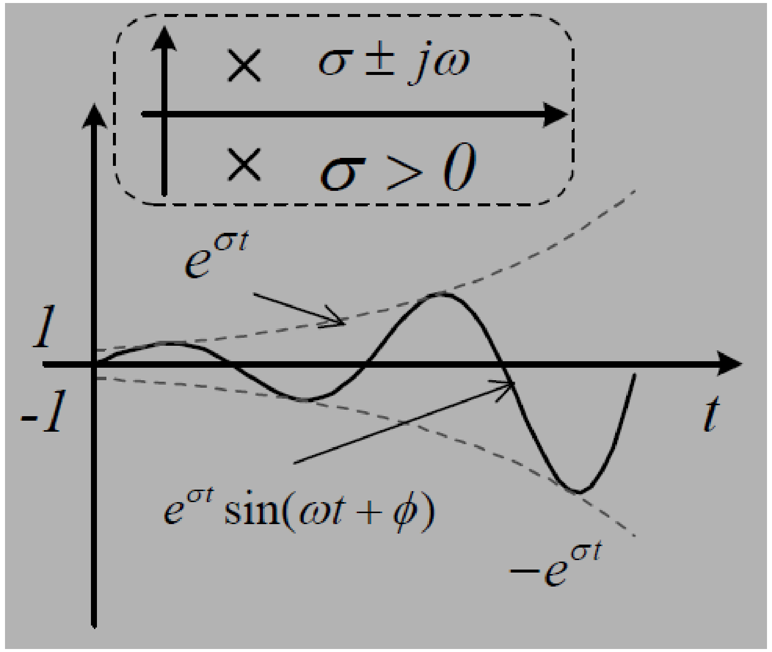
Transient Response and Pole Locations

- Complex poles always in conjugate pairs:

$$-p_{1,2} = \sigma \pm j\omega \qquad 3 \pm 4j \quad -2 \pm 3j$$

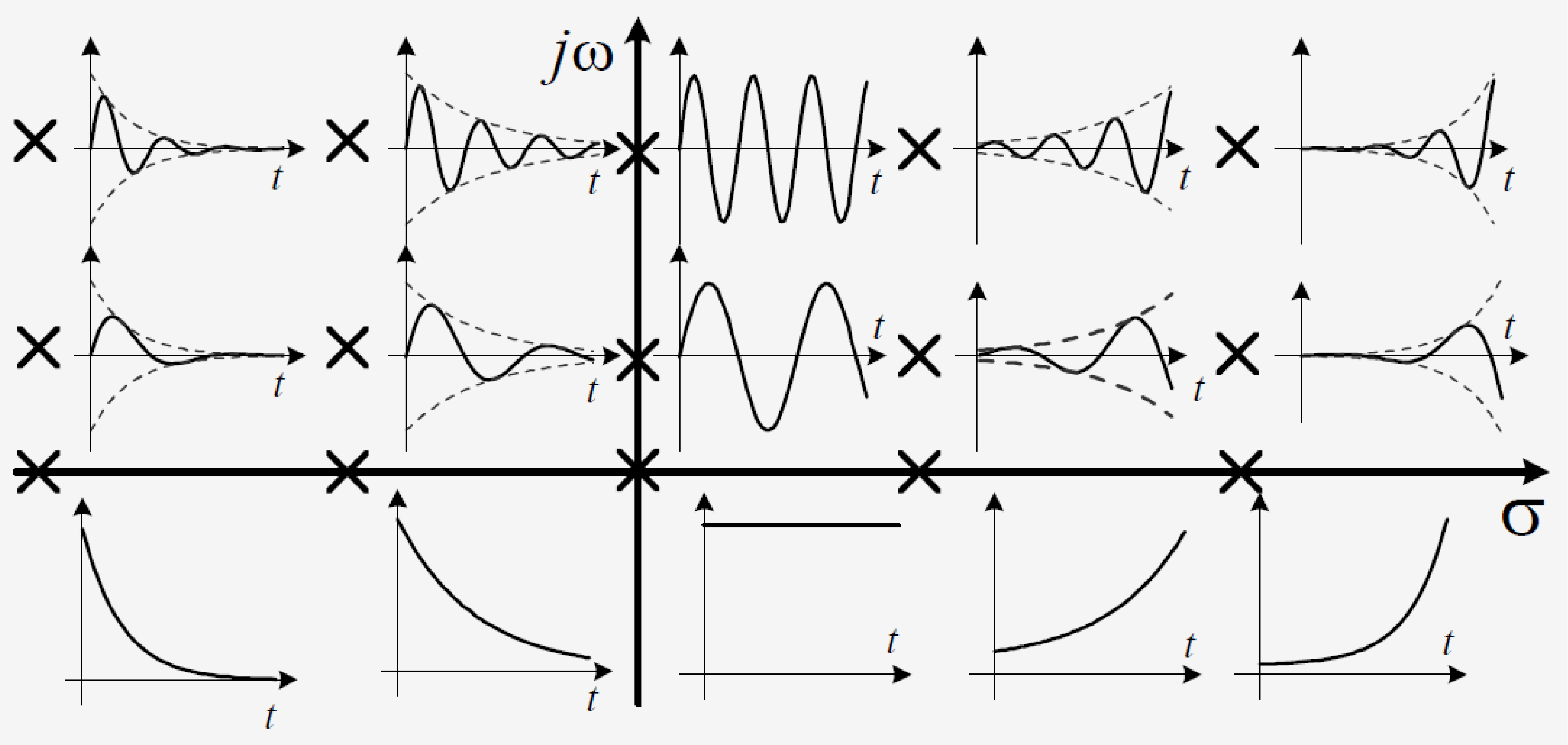
- Combined response: $Ae^{\sigma t}\sin(\omega t + \phi)$





- The real part σ determines the decay (or growth) rate. The imaginary part ω determines the oscillation frequency.

Pole locations and transient responses



$pole = \sigma \pm j\omega$

$\sigma < 0 \quad \Rightarrow \quad y_0(t) \rightarrow 0$ when $t \rightarrow \infty$

$\sigma > 0 \quad \Rightarrow \quad y_0(t) \rightarrow \infty$ when $t \rightarrow \infty$

Unit Step Response and Stability

- Unit step input: $u(t) = 1, \quad U(s) = \frac{1}{s}$
- The Laplace transform of the output is

$$Y(s) = G(s)U(s) = \frac{N(s)}{s(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- Partial fraction expansion:
  - Assume that the system has distinct poles (real or complex conjugate). i.e. $p_i \neq p_j$ (when $i \neq j$)

$$Y(s) = \frac{A_0}{s} + \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \cdots + \frac{A_n}{s+p_n}$$

Unit Step Response

- Partial fraction expansion coefficients:

$$A_0 = s \cdot Y(s)\big|_{s=0} \qquad A_i = (s+p_i)\cdot Y(s)\big|_{s=-p_i} \quad i = 1,\cdots,n$$

cover-up rule

- Output time response:

$$y(t) = \mathcal{L}^{-1}\left(Y(s)\right)$$

$$= \mathcal{L}^{-1}\left(\frac{A_0}{s} + \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \cdots + \frac{A_n}{s+p_n}\right)$$

$$= \left(A_0\right) + \left(A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \cdots + A_n e^{-p_n t}\right)$$

Forced and Transient Responses

$$y(t) = (A_0) + \left(A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \cdots + A_n e^{-p_n t}\right)$$

$y_f(t)$: forced response due to input

$y_0(t)$: transient response (natural response) due to the nature of the system

- The characteristics of the transient response is determined by the system transfer function!

Transfer function		Transient response
$\dfrac{A_i}{s+p_i}$	$\Rightarrow$	$A_i e^{-p_i t}$

Stability and Pole Location

- For a system to be useful, the transient response must decay to zero.

$$y_0(t) \to 0 \;\text{ when }\; t \to \infty \quad \Rightarrow \quad y(t) \to y_f(t) \;\text{ when }\; t \to \infty$$

- A system is **stable** if the transient response decays to zero and is **unstable** if this response grows.

$$\text{stable} \quad \Leftrightarrow \quad y_0(t) \to 0 \;\text{ when }\; t \to \infty$$

- The characteristics of the transient response is determined by the pole positions. e.g.

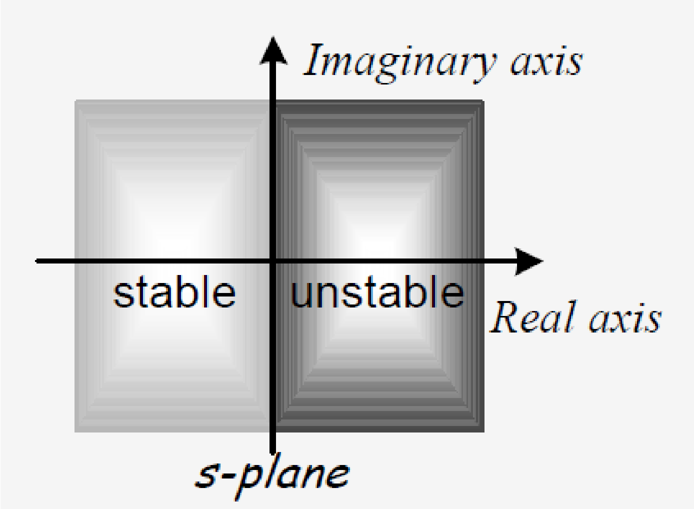
$$\text{pole } -2 \quad \Rightarrow \quad e^{-2t} \to 0 \;\text{ when }\; t \to \infty$$

$$\text{pole } \;\; 2 \quad \Rightarrow \quad e^{2t} \to \infty \;\text{ when }\; t \to \infty$$

Stability Theorem

- A system is stable *if and only if* all system poles lie in the left hand side of *s*-plane, i.e.

$$\mathrm{Re}(poles) = \mathrm{Re}(-p_i) < 0 \quad i = 1, 2, \cdots n$$



- To verify the system stability, we need to find all system poles.

Check Stability Using Poles

- **Poles:** The solutions (roots) of the system characteristic equation (CE).

$$G(s)=\frac{N(s)}{a_n s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

- CE: $$a_n s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+a_0=0$$

- $n=2$: $$a_2 s^2 + a_1 s + a_0 = 0$$

  - *Poles*: $$s_{1,2}=\frac{-a_1\pm\sqrt{a_1^2-4a_2a_0}}{2a_2}$$

CE: $s^2+4s+8=0$ poles: $s_{1,2}=-2\pm j2$

$\mathrm{Re}\{s_{1,2}\}<0$ stable system

CE: $s^2+2s-8=0$ poles: $s_1=-4,\ s_2=2$

$\mathrm{Re}\{s_2\}=2>0$ unstable system

Poles of systems with order larger than 3

- $n \geq 3$, e.g. $s^3 + 4s^2 + 3s + 5 = 0$ Poles?
- It is not easy to find the solutions for the CE with order larger than 2.

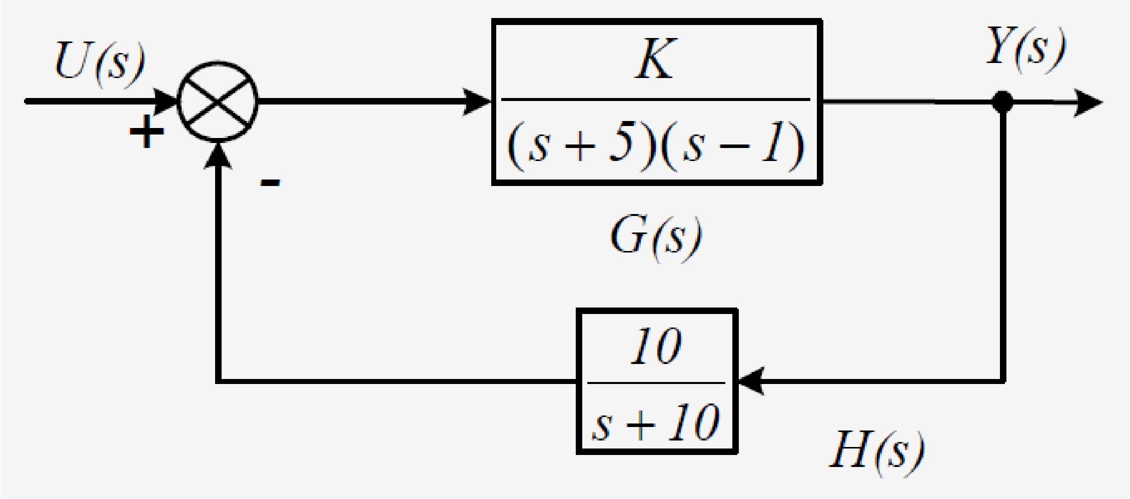
Poles found using MATLAB: -3.552, $-0.224 \pm 1.165j$

Problem can be solved, but not easy.

- Can we check the system stability without the need of finding poles?
  - Using Routh-Hurwitz Stability Criterion!

Example

- For what values of gain K, if any, is the system show below stable?



Closed loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K(s+10)}{s^3 + 14s^2 + 35s + (10K - 50)}$$

CE: $s^3 + 14s^2 + 35s + (10K - 50) = 0$

Example

CE: $s^3 + 14s^2 + 35s + (10K - 50) = 0$

$K=4$ Poles found using MATLAB: -10.61, -3.63, 0.25

`roots([1 14 35 -10])` MATLAB command

$K=10$ Poles found using MATLAB: -11.29, -$1.35 \pm 1.61j$

$K=60$ Poles found using MATLAB: -14.25, $0.13 \pm 6.21j$

How to find the stability region for K? Not possible to solve this problem so far.

- Can we check the system stability without the need of finding poles?
  - Using Routh-Hurwitz Stability Criterion!

Routh-Hurwitz Stability Criterion

- Step 1: Write down the characteristic equation (CE).

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- Step 2: Check all coefficients.
  - If all coefficients are not in the same sign (some positive and some negative) or one or more coefficients missing (zero), the system is unstable. Stop here.

$$Unstable: \quad s^3 + 4s^2 - 3s + 8; \quad s^3 + 4s^2 + 8$$

  - If all coefficients are in the same sign (all positive or all negative) and no missing coefficients, the system can be stable or unstable. Go to Step 3.

$$Unsure: \quad s^3 + 4s^2 + 3s + 8$$

Routh-Hurwitz Stability Criterion

- Step 3: Construct Routh Table.
- Step 4: Check the signs of the first column of the Routh Table.
  - The *necessary and sufficient condition* for stability is that there is *no changes of sign* in the elements of the 1st column of Routh Table (all positive or all negative).
  - The number of these sign changes is equal to the number of poles with positive real parts.

Routh Table

$$CE: \quad a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	0
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	0
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	0

Example: $CE: \quad s^4 + 5s^3 + 7s^2 + 3s + 5 = 0$

s^4	1	7	5		+
s^3	5	3	0		+
s^2	$\frac{5 \times 7 - 1 \times 3}{5} = 6.4$	$\frac{5 \times 5 - 1 \times 0}{5} = 5$	0		+
s^1	$\frac{6.4 \times 3 - 5 \times 5}{6.4} = -0.9$	0	0		-
s^0	$\frac{-0.9 \times 5 - 6.4 \times 0}{-0.9} = 5$	0	0		+

Example: $CE: \quad s^4 + 5s^3 + 7s^2 + 3s + 5 = 0$

There are two changes in sign (+ to - and - to +) in the first column, hence the system is unstable and there are two poles with positive real parts.

Verification:

Poles are: $-2.55 \pm 0.546j,$

$0.055 \pm 0.854j$

Unstable system