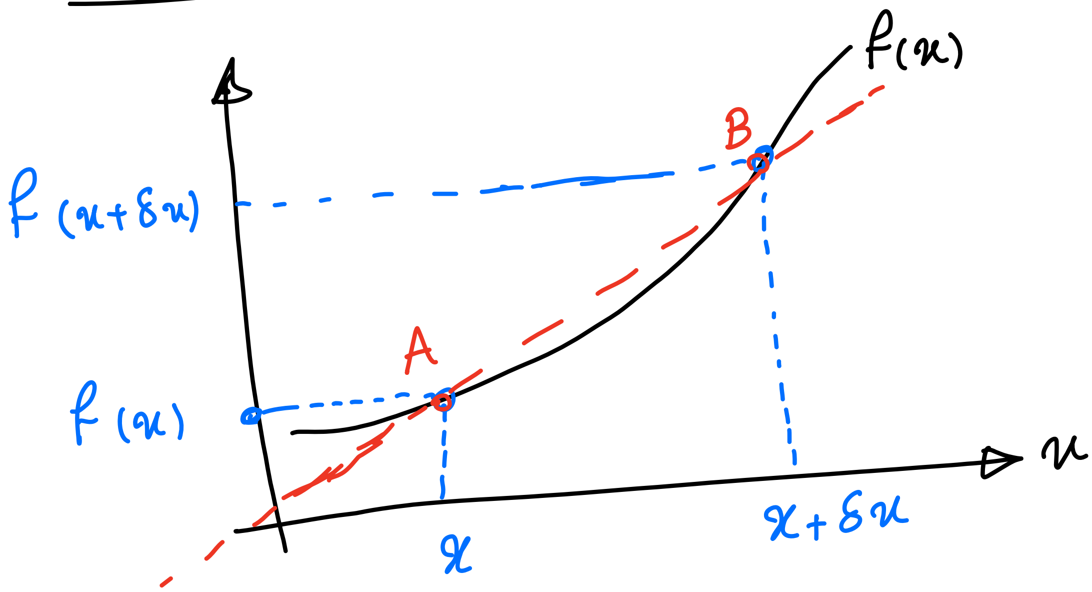
Derivatives and applications



AB is the secant

$$\text{slope of } AB = \frac{f(x+\delta x) - f(x)}{(x+\delta x) - x}$$

$$= \frac{f(x+\delta x) - f(x)}{\delta x}$$

In the limit when $A \to B \Rightarrow \delta x \to 0$

The secant of $AB \longrightarrow$ tangent at A

$$\lim \text{ Slope of } AB: \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Definition of the derivative

Notations:

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{df}{dx} = f'(x)$$

Derivative of f with respect to x

Common Derivatives

$$\frac{d}{dx} c = 0$$

constant

$$\frac{d}{dx} x = 1$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \log x = 1/x$$

$$\frac{d}{dx} a^x = a^x \log a$$

Note: log means log with base "e"

$\log_e$ or $\ln$

↓

Natural logarithim

If you need to write log base 10

use $\log_{10}$

product rule

$f(x)$ and $g(x)$

find $\frac{d}{dx}(f(x)\,g(x))$

$$= f'(x)\,g(x) + f(x)\,g'(x)$$

Example

$$\frac{d}{dx}(x^2 \sin x)$$

$$= \frac{d(x^2)}{dx}\sin x + x^2 \frac{d(\sin x)}{dx}$$

$$= 2x \sin x + x^2 \cos x$$

Example

$$\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{d}{dx}\left(x^{-1} e^x\right)$$

$$= (-1)x^{-2} e^x + x^{-1} e^x$$

Chain Rule

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}$$

Example

$f(x) = \sin x$ $\qquad$ $g(x) = x^2$

$$f(g(x)) = f(x^2) = \sin x^2$$

$f(g) = \sin g$ $\quad$ where $\quad$ $g(x) = x^2$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = (\cos g)\ (2x)$$

$g(x) = x^2$

$$\frac{df}{dx} = \cos(x^2).(2x)$$

Example

$f(x) = \sin x \qquad g(x) = e^x$

$$\frac{d\,f(g)}{dx} = ?$$

$f(g(x)) = \sin g = \sin(e^x)$

$$\frac{d}{dx} = \frac{df}{dg}\,\frac{dg}{dx} = (\cos g)\,(e^x)$$

$$= (\cos e^x)\,(e^x)$$

Chain rule

$$\frac{d}{dx}\left\{ f(g(h(x))) \right\} = \frac{df}{dg}\,\frac{dg}{dh}\,\frac{dh}{dx}$$

Example

$$\frac{d}{dx}\left[\sin\left(2e^{x^2}\right)\right]$$

$f \rightarrow \sin g$

$g \rightarrow 2e^{h}$

$h \rightarrow x^2$

$$f(g(h(x))) = \sin g$$

$$= \sin(2e^{h}) = \sin(2e^{x^2})$$

$$\frac{df}{dx} = ?$$

$\cos g$

$2e^{h}$

$2x$

$$\frac{df}{dx} = \frac{df}{dg}\,\frac{dg}{dh}\,\frac{dh}{dx}$$

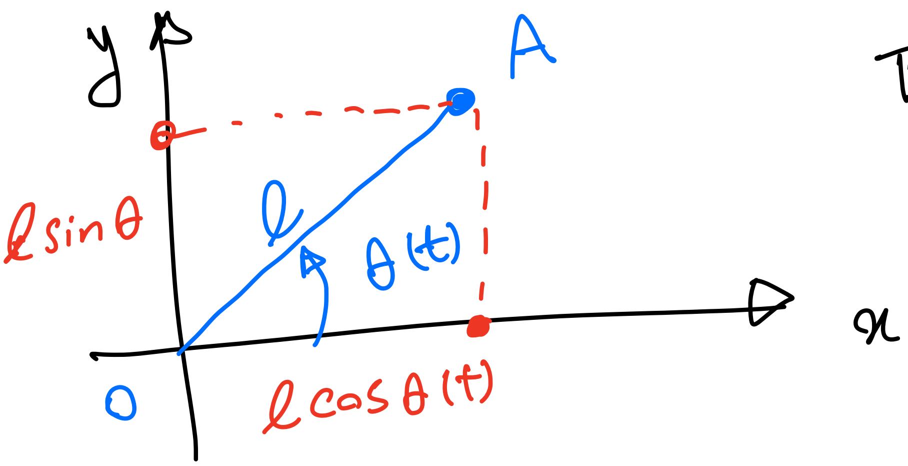
$$\frac{df}{dx} = (\cos g)\,(2e^{h})\,(2x)$$

$$\left\{\begin{array}{l} g \rightarrow 2e^{h} \\ h \rightarrow x^{2} \end{array}\right.$$

$$= \left[\cos\left(2e^{x^2}\right)\right]\left[2e^{x^2}\right] 2x$$

$$= 4xe^{x^2}\cos\left(2e^{x^2}\right)$$

Example



OA is rotating about point O. The angular displacement of OA is $\theta(t)$ at time t.

Notation: $\dot{\theta} = \dfrac{d\theta}{dt}$

Find the velocity of OA in x direction.

velocity in x direction $\longrightarrow V_x$

$$V_x = \frac{dx_A}{dt} = \frac{d}{dt}(l \cos\theta(t))$$

$$= l \frac{d}{dt} \cos\theta(t)$$

$$= l \frac{d}{d\theta} \cos\theta(t) \frac{d\theta}{dt}$$

$$= l(-\sin\theta) \underbrace{\frac{d\theta}{dt}}_{\dot{\theta}}$$

$$= l(-\sin\theta)\,\dot{\theta}$$

Calculate the acceleration of A in x direction.

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt}$$

$$a_x = \frac{d}{dt}\left[(-l\sin\theta)(\dot{\theta})\right]$$

product rule:

$$a_x = -l\left[\frac{d}{dt}(\sin\theta)\,(\dot{\theta}) + (\sin\theta)\frac{d}{dt}\dot{\theta}\right]$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2}$$

chain rule:

$$a_x = -l\left[\frac{d}{d\theta}(\sin\theta)\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}}(\dot{\theta}) + (\sin\theta)\,\ddot{\theta}\right]$$

$$a_x = -l\left[\cos\theta\,(\dot{\theta})^2 + (\sin\theta)\,\ddot{\theta}\right]$$

$\dot{\theta} \longrightarrow$ angular velocity

$\ddot{\theta} \longrightarrow$ angular acceleration