Instrumentation and Controls

ETM 3301

Lecture 12

Instructor

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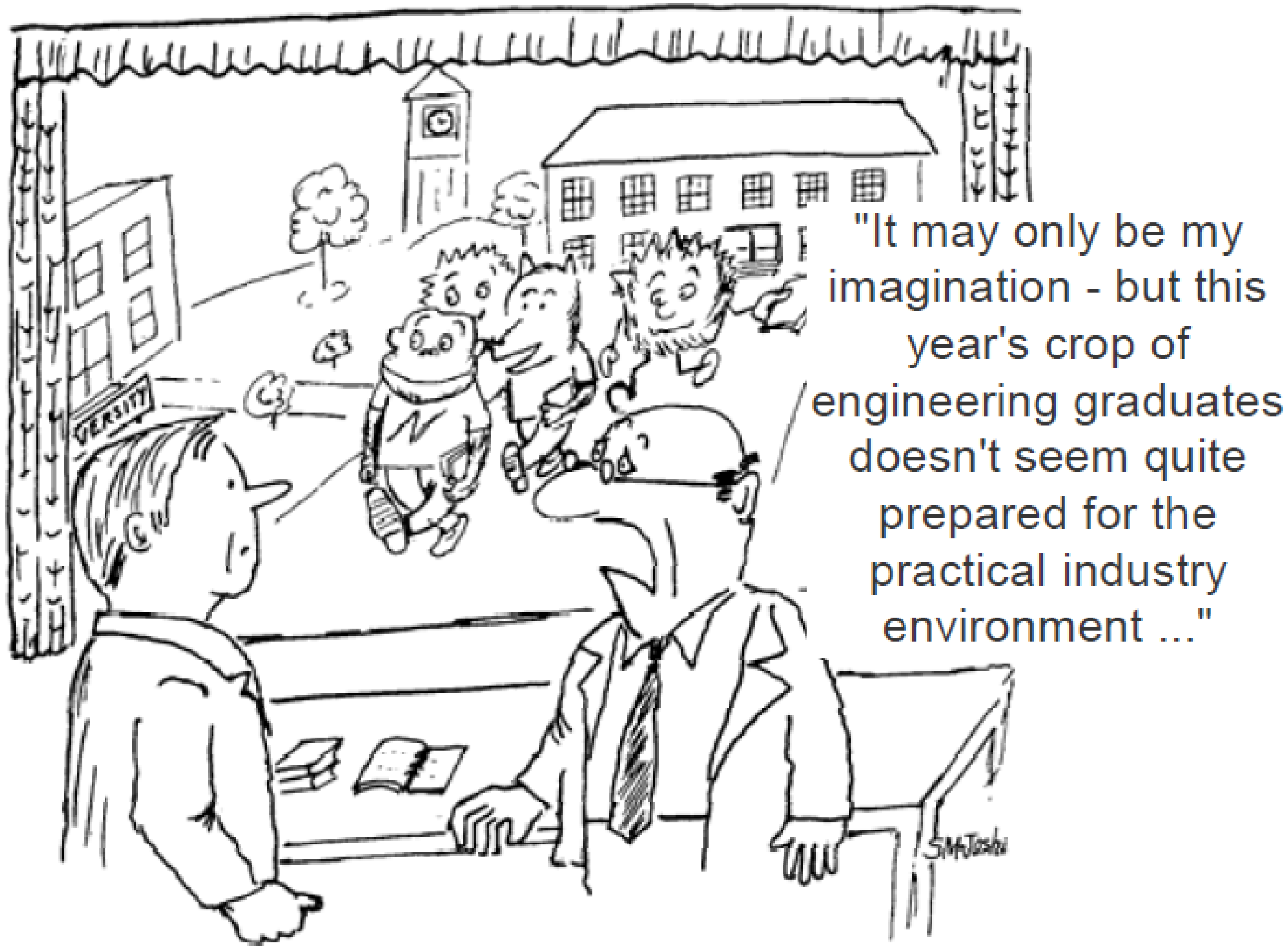
Experimental Determination of 1st and 2nd Order Transfer Functions



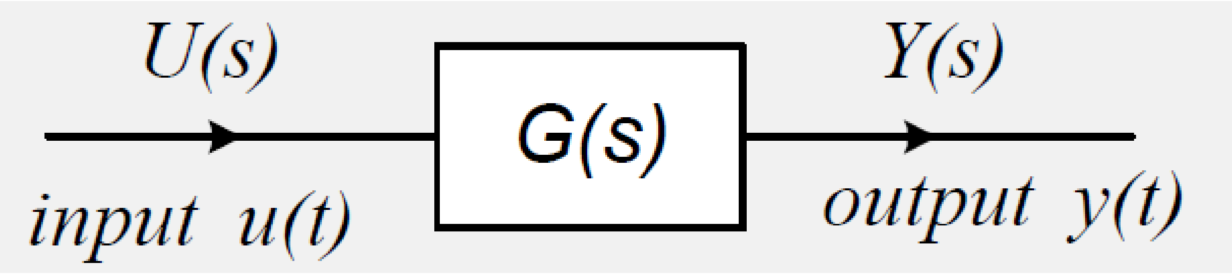
- Find 1st and 2nd order system transfer functions through a number of experimental tests.
- Gain practical experience of analysing a typical system often found in real world.

Transfer Function and Block Diagram

- A simple block diagram illustrates a system's input, output and transfer function. The signal flows from input, through the transfer function, to output.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}$$

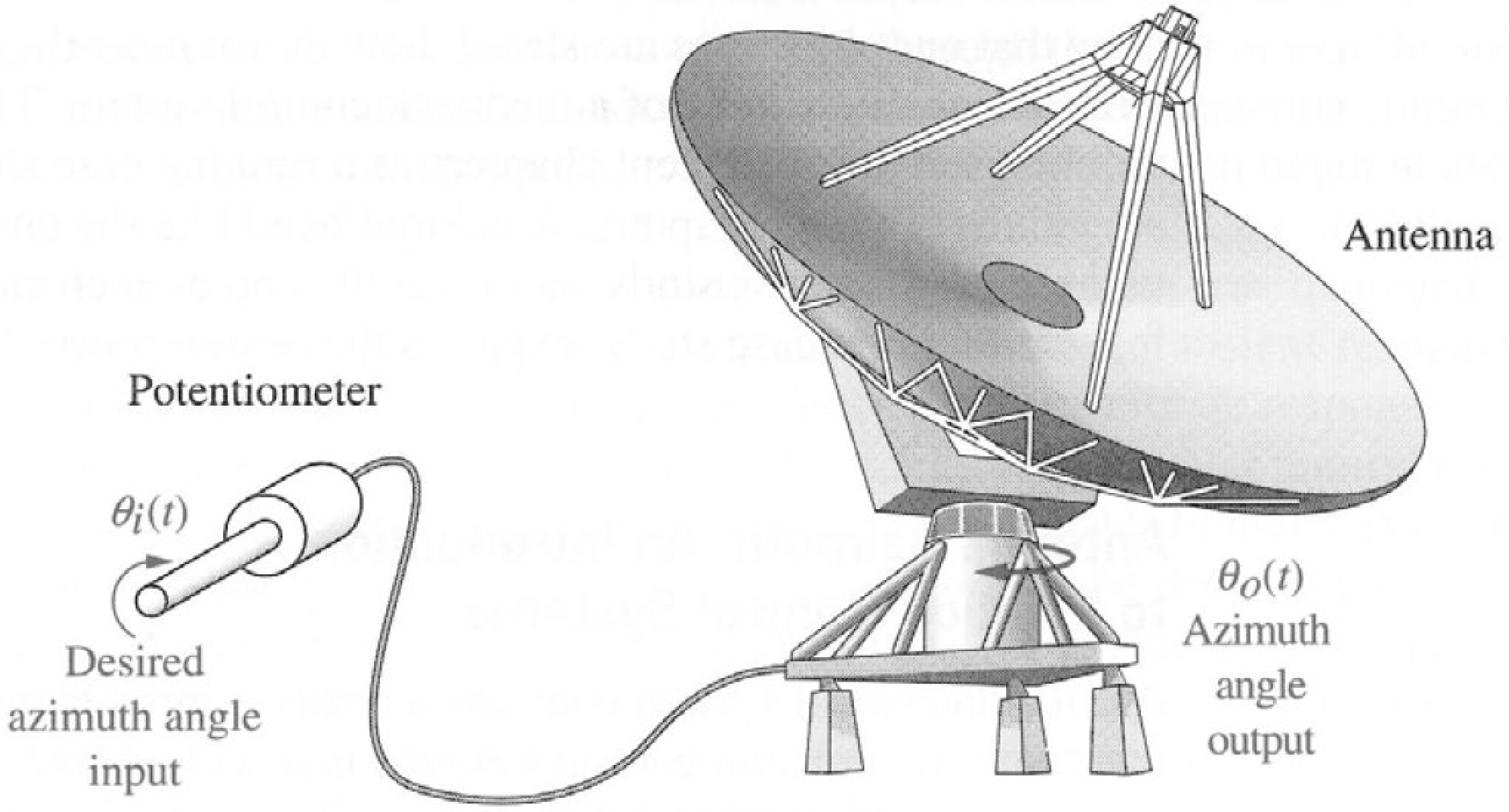
Input	Output
Action	Consequence
Cause	Effect
Command	Response



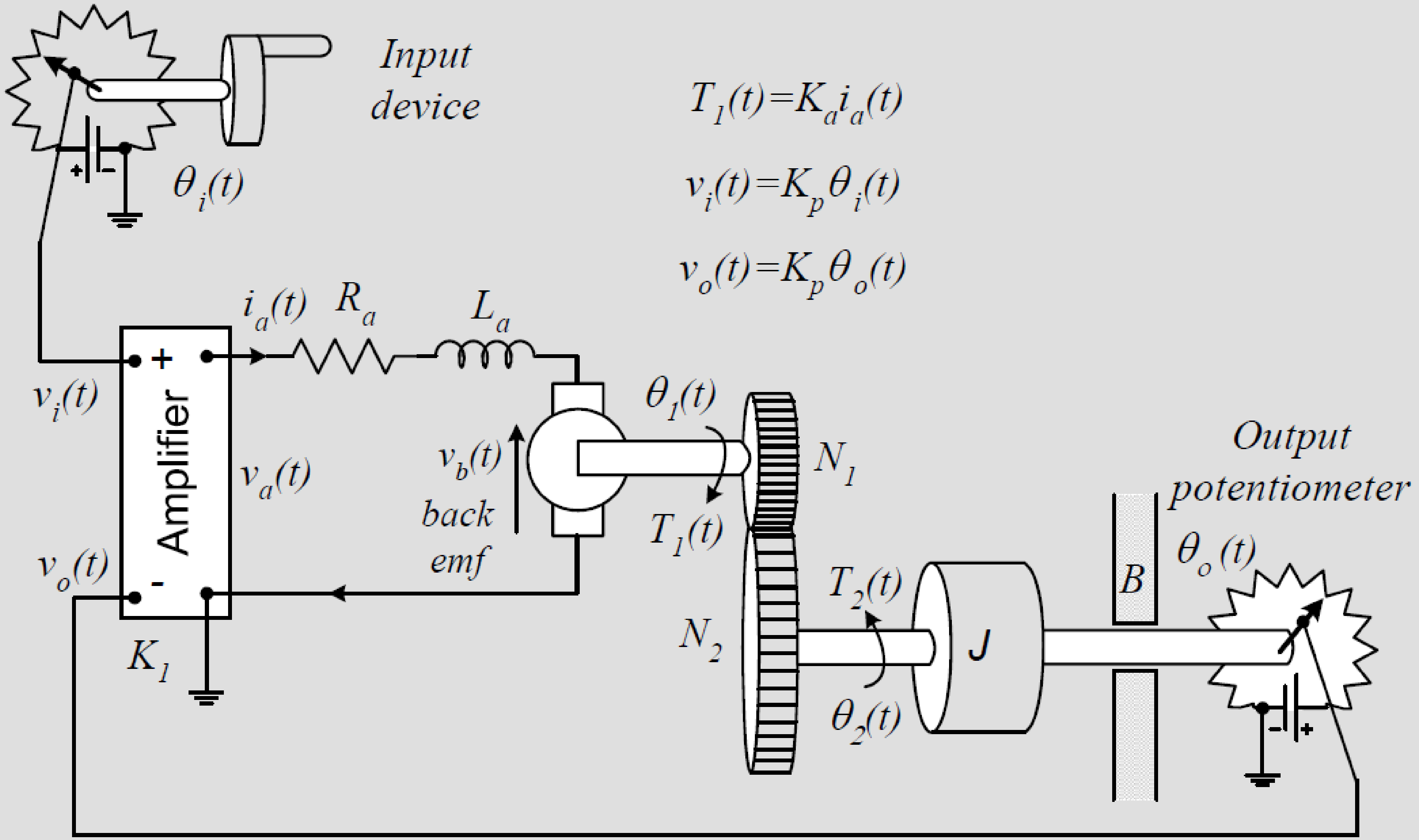
How to obtain the transfer function?

Servo System Example

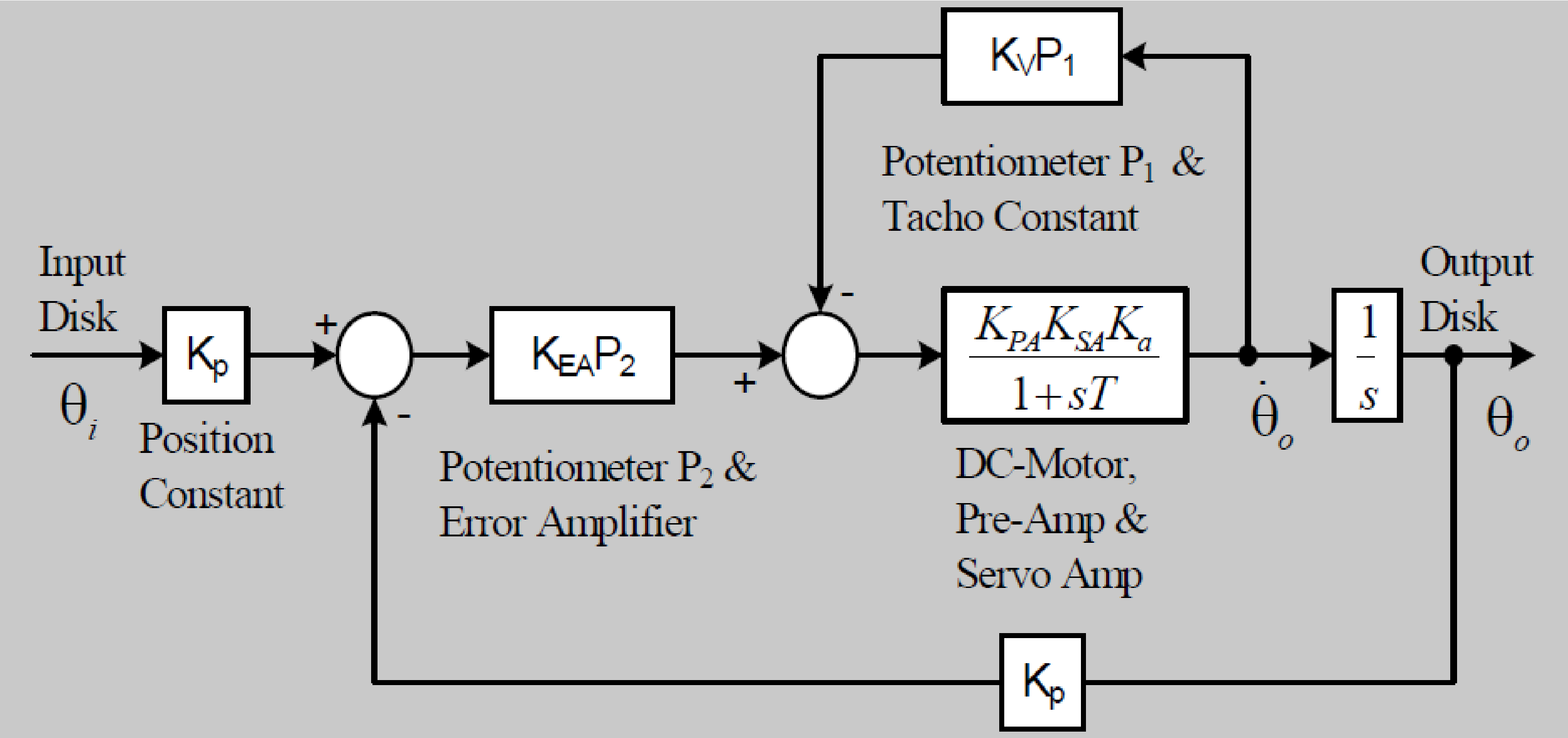
- Antenna for tracking star/aircraft position

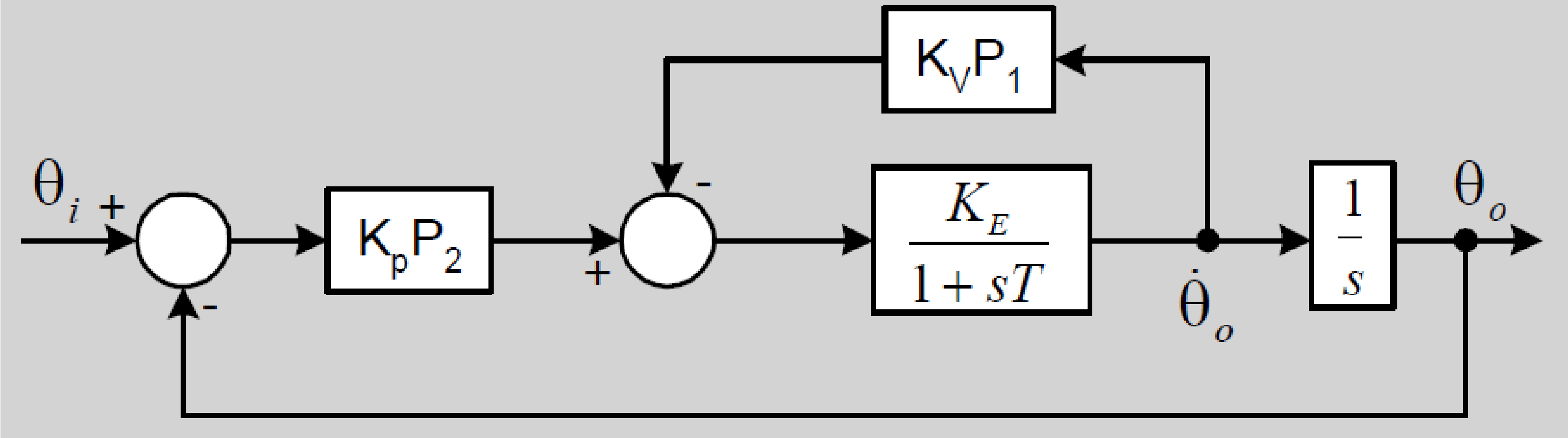


Servo System Schematic Diagram



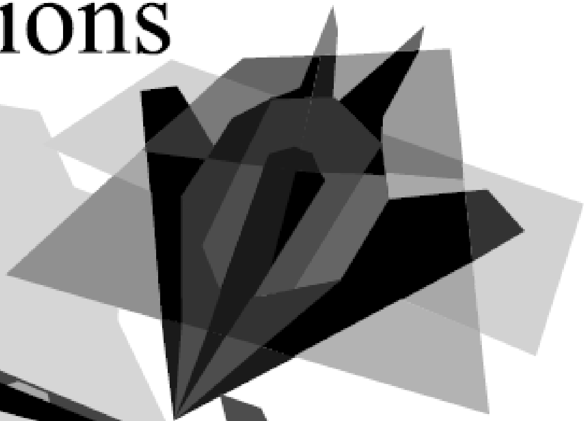
Servo System Block Diagram

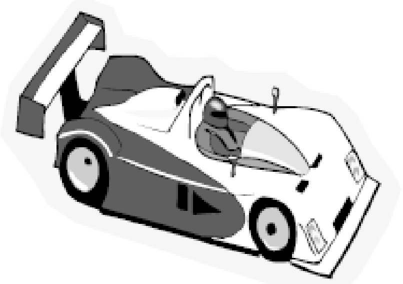




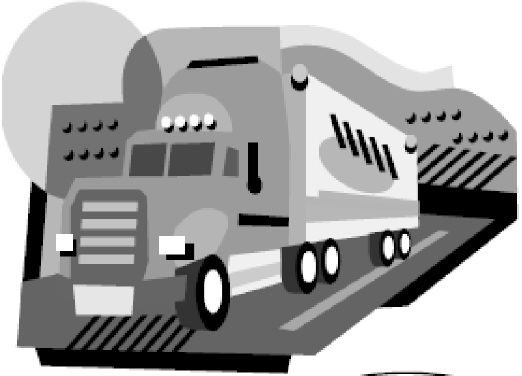
Servo System Applications

Flight-by-Wire fight control system



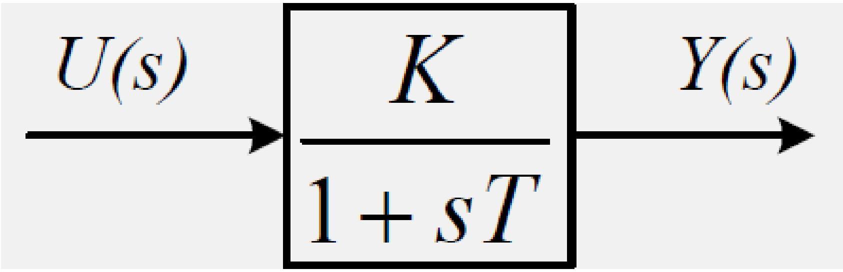


Drive-by-Wire steering control system

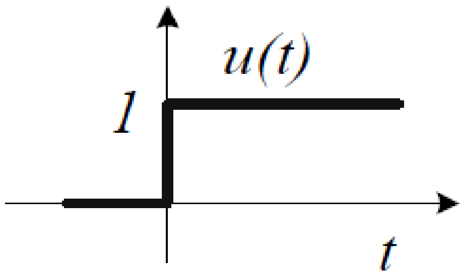


1st order system <u>unit step</u> response

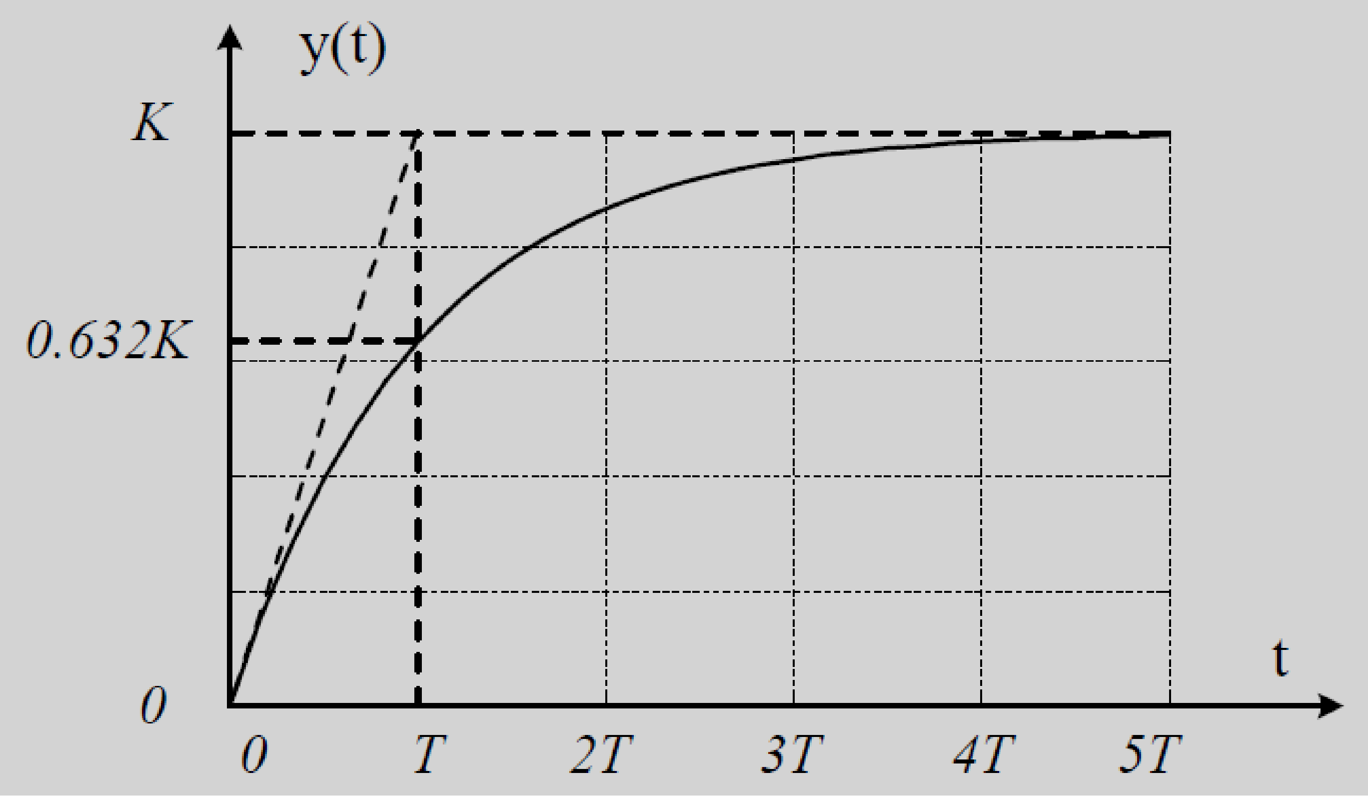
$$u(t) = 1$$



$$y(t) = K\left(1 - e^{-\frac{t}{T}}\right)$$

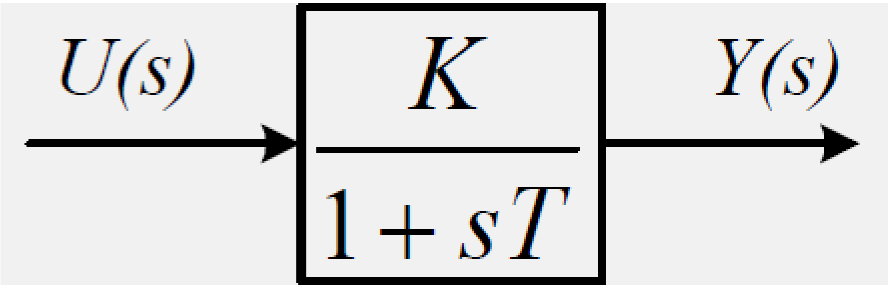


- Time constant T



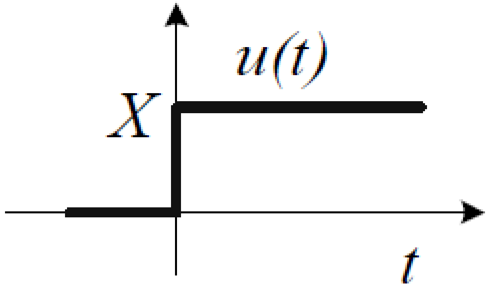
1st order system <u>step</u> response

$$u(t) = X$$



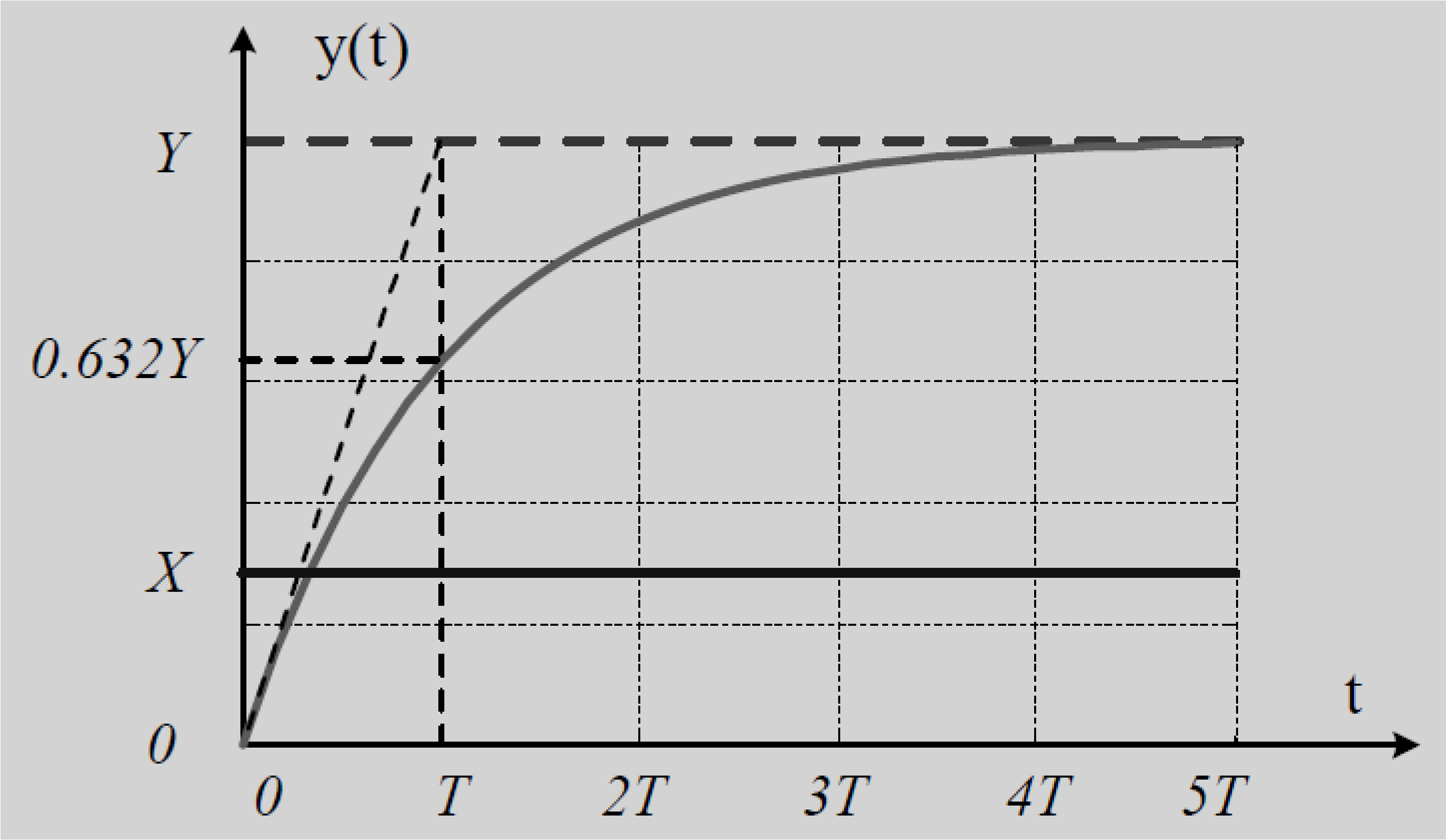
$$y(t) = Y\left(1 - e^{-\frac{t}{T}}\right)$$

$$Y = KX$$



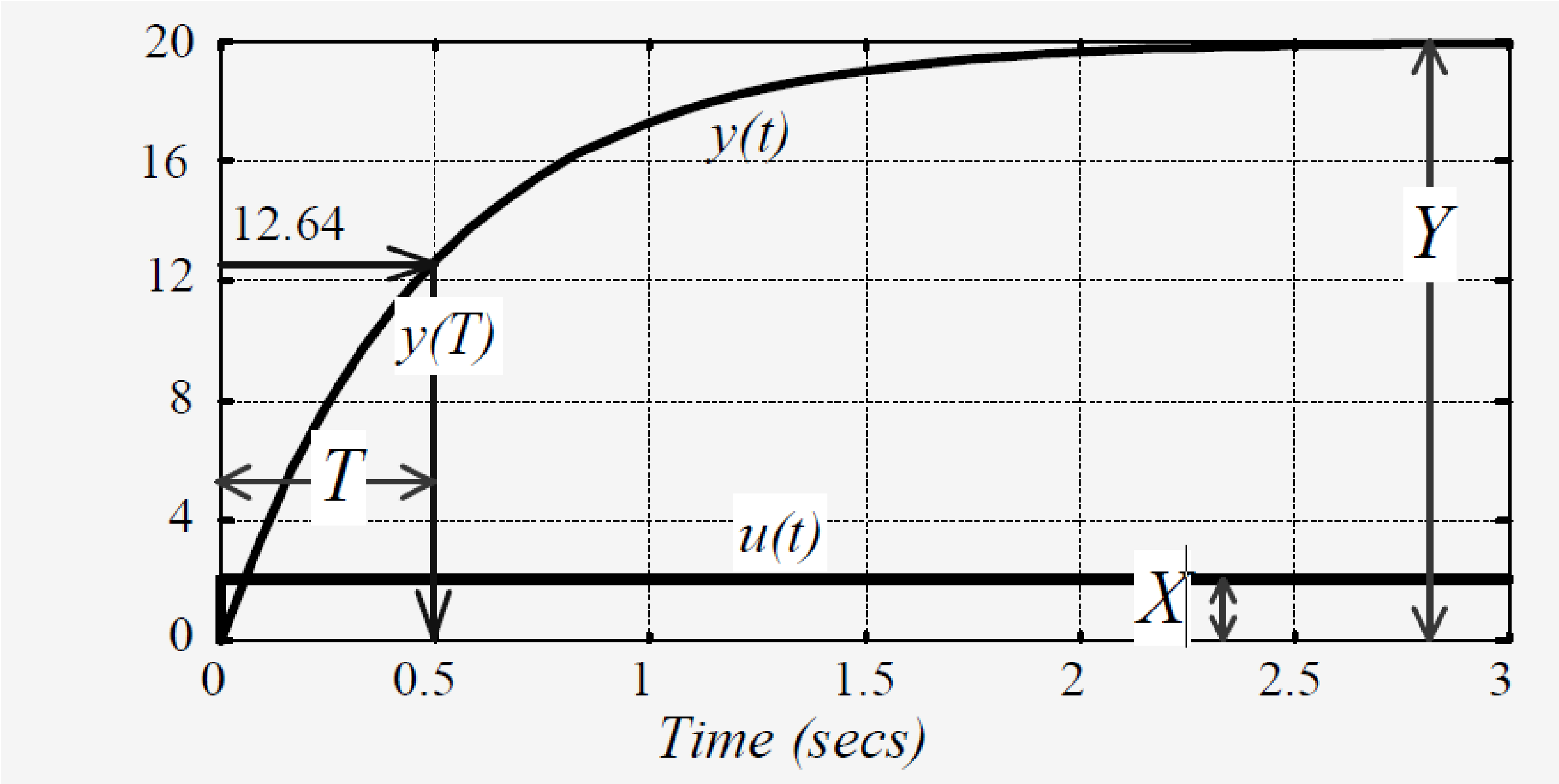
$$y_{ss} = Y = KX$$

$$y(T) = 0.632Y$$



1st order transfer function via experiment

- With a step input, we can measure the time constant and steady-state value, from which the transfer function can be calculated.



1st order transfer function via experiment: Result

- Find K: The step response reaches a final value of

$$y_{ss} = Y = KX = 20$$

Input magnitude $X=2$

$$K = \frac{Y}{X} = \frac{20}{2} = 10$$

- The time constant, T, is the time for the response to reach 0.632 of its final value.

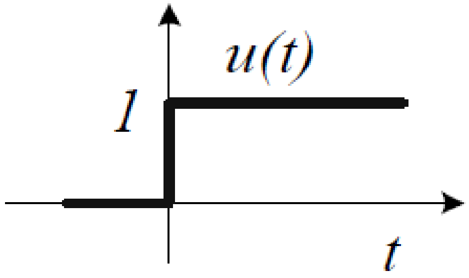
$$y(T) = 0.632 \times 20 = 12.64 \quad \Rightarrow \quad T \approx 0.5s$$

$$G(s) = \frac{K}{1+sT} = \frac{10}{1+0.5s} = \frac{20}{s+2}$$

2nd order system unit step response

$$u(t) = 1$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



ζ *damping ratio*

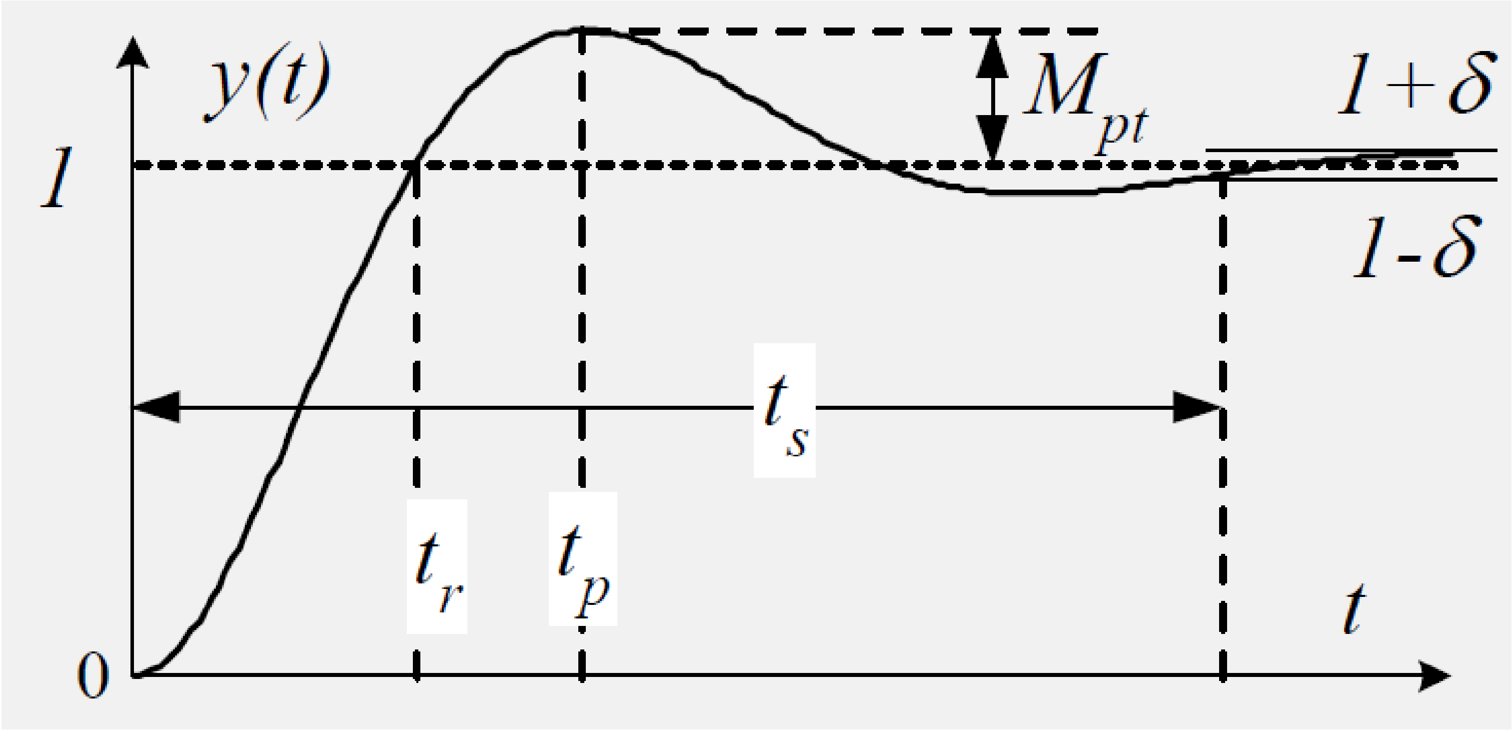
ω_n *undamped natural frequency*

- Response to a unit step input

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$$

$$\phi = \cos^{-1}\zeta; \quad 0 < \zeta < 1$$

2nd order system **unit step** response



M_{pt} — *overshoot*

t_s — *settling time*

Percentage overshoot (PO): $PO = M_{pt} \times 100\%$

2nd order system unit step response Performance measures

Peak time $$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Damped natural frequency: $$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

Overshoot: $$M_{pt} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

Percentage overshoot: $$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

2nd order system unit step response Performance measures

Settling time:

Settling within *5%* error band

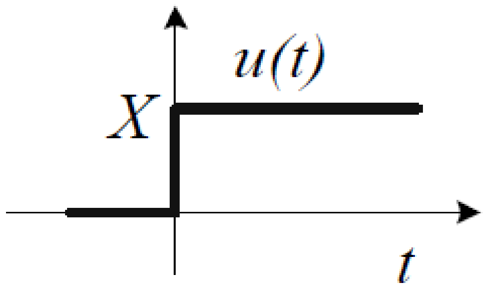
$$t_s = \frac{3}{\zeta \omega_n}$$

Settling within *2%* error band

$$t_s = \frac{4}{\zeta \omega_n}$$

2nd order system <u>step</u> response

$$u(t) = X$$



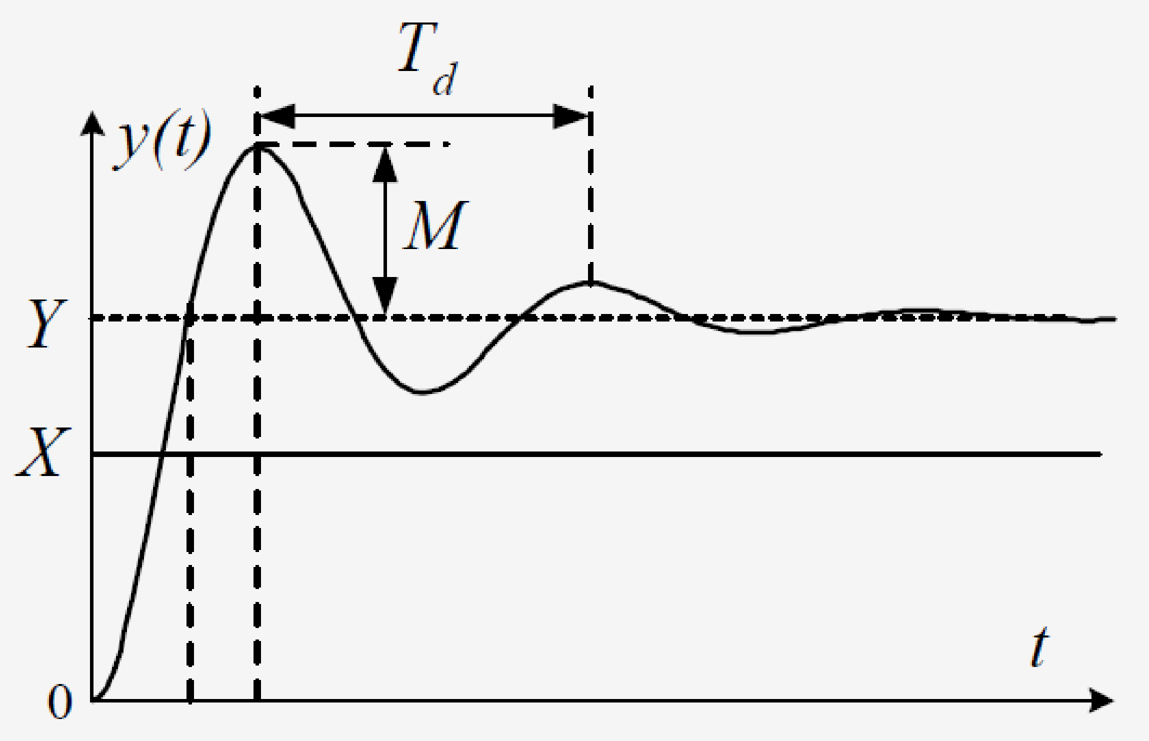
$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Response

$$y(t) = Y\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)\right]$$

$$Y = kX, \quad \phi = \cos^{-1}\zeta, \quad 0 < \zeta < 1$$

2nd order system <u>step</u> response



$$PO = \frac{M}{Y} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$k = \frac{Y}{X}$$

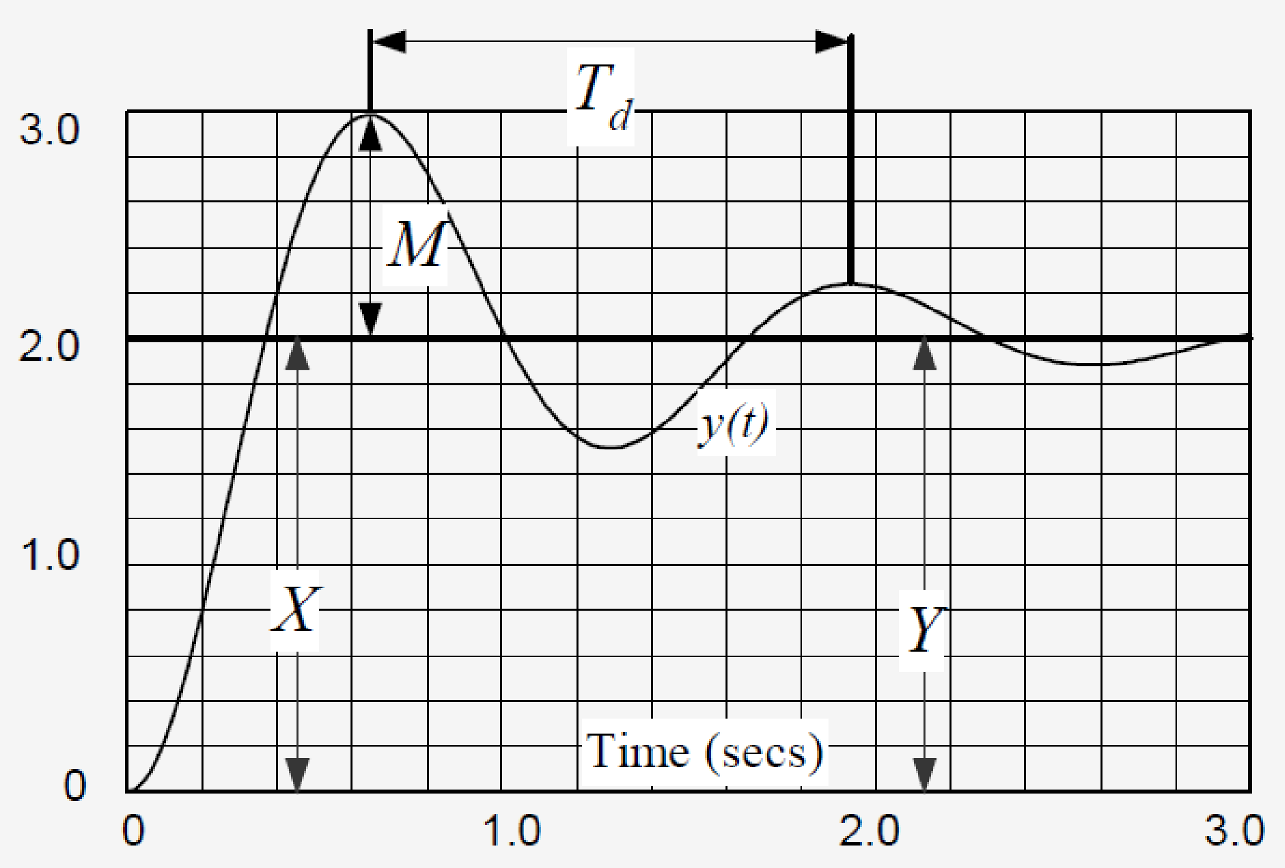
- *Damped oscillation:*

Nature frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Period: $T_d = \dfrac{2\pi}{\omega_d}$

2nd order transfer function via experiment, 1

- We can measure the response curve for percentage overshoot and damped oscillation period, from which we can found damping ratio (ζ) and nature frequency (ω_n).



2nd order transfer function via experiment, 2

- Step Input: $u(t)=X=2$
- Overshoot: $M \approx 1$
- Final value: $Y=2$
- Percentage Overshoot: $PO = \frac{M}{Y} = \frac{1}{2} = 0.5 = 50\%$
- Damped oscillation period: $T_d \approx 1.3s$

$$k = \frac{Y}{X} = \frac{2}{2} = 1$$

2nd order transfer function via experiment, 3

$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad \Rightarrow \quad \ln(PO) = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \quad \left(\ln(PO)\right)^2 = \frac{\pi^2\zeta^2}{1-\zeta^2}$$

$$\Rightarrow \quad \left(\ln(PO)\right)^2 - \left(\ln(PO)\right)^2\zeta^2 = \pi^2\zeta^2$$

$$\Rightarrow \quad \zeta^2 = \frac{\left(\ln(PO)\right)^2}{\pi^2 + \left(\ln(PO)\right)^2}$$

$$\Rightarrow \quad \zeta = \frac{\left|\ln(PO)\right|}{\sqrt{\pi^2 + \left(\ln(PO)\right)^2}} = \frac{\left|\ln(0.5)\right|}{\sqrt{\pi^2 + \left(\ln(0.5)\right)^2}} = 0.215$$

2nd order transfer function via experiment, 4

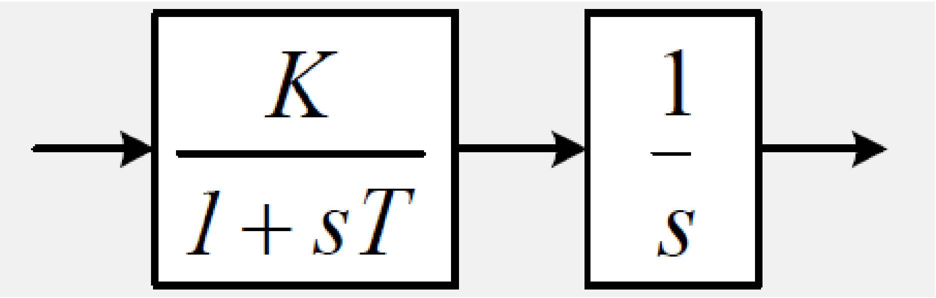
$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{1.3} = 4.83 rad/s$$

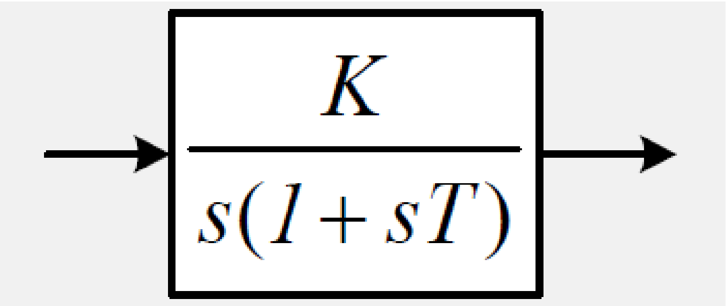
$$\omega_d = \omega_n\sqrt{1-\zeta^2} \;\Rightarrow\; \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 4.95 rad/s$$

Finally:

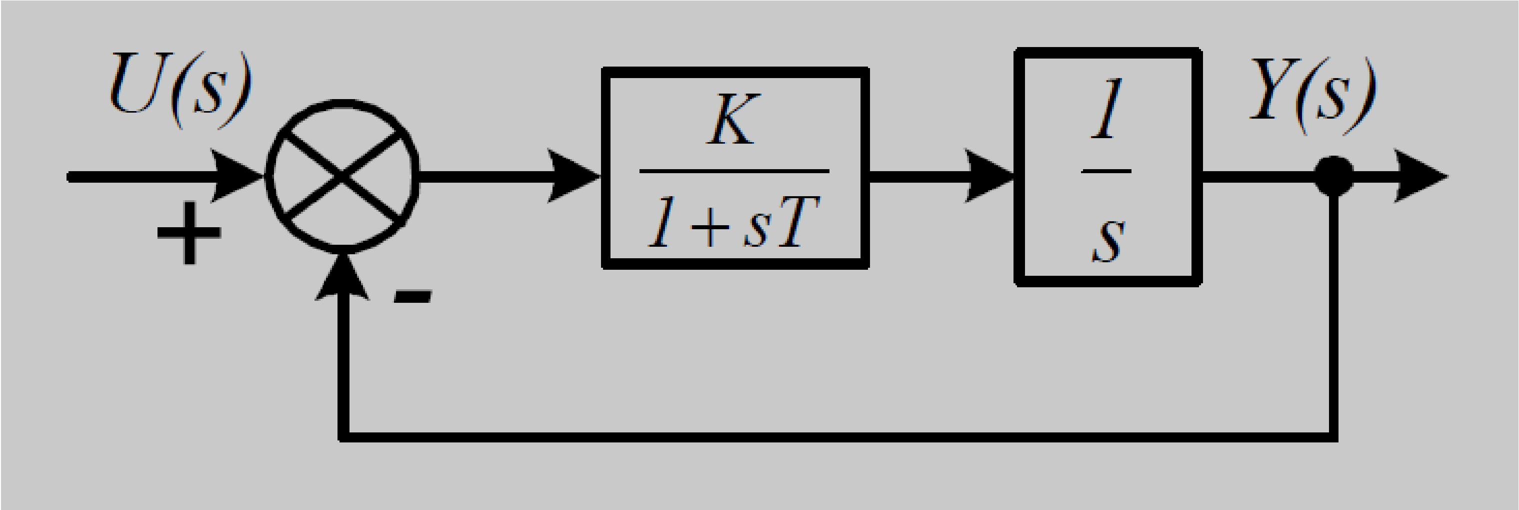
$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{24.5}{s^2 + 2.13s + 24.5}$$

Forming a 2nd order system via two 1st order systems

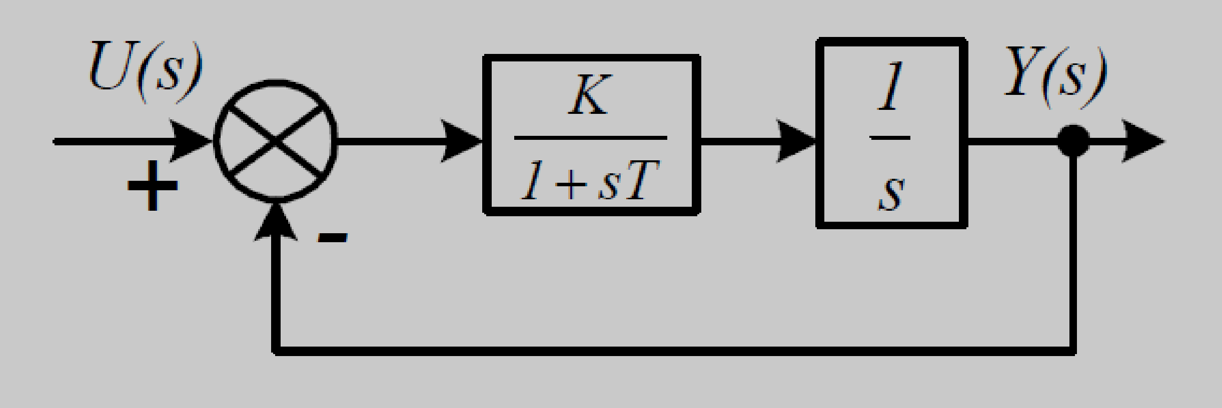




Experiment: $K=10$ $T=0.5$



2nd order system TF



$$G(s)=\frac{Y(s)}{U(s)}=\frac{\dfrac{K}{s(1+sT)}}{1+\dfrac{K}{s(1+sT)}}=\frac{K}{Ts^2+s+K}=\frac{\left(K/T\right)}{s^2+\left(1/T\right)s+\left(K/T\right)}$$

Comparing with standard 2nd system TF:

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$s^2+\left(1/T\right)s+\left(K/T\right)=s^2+2\zeta\omega_n s+\omega_n^2$$

2^{nd} order system TF parameters

$$s^2 + \left(\frac{1}{T}\right)s + \left(\frac{K}{T}\right) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\omega_n^2 = \frac{K}{T} \quad \Rightarrow \quad \omega_n = \sqrt{\frac{K}{T}} \approx 4.47$$

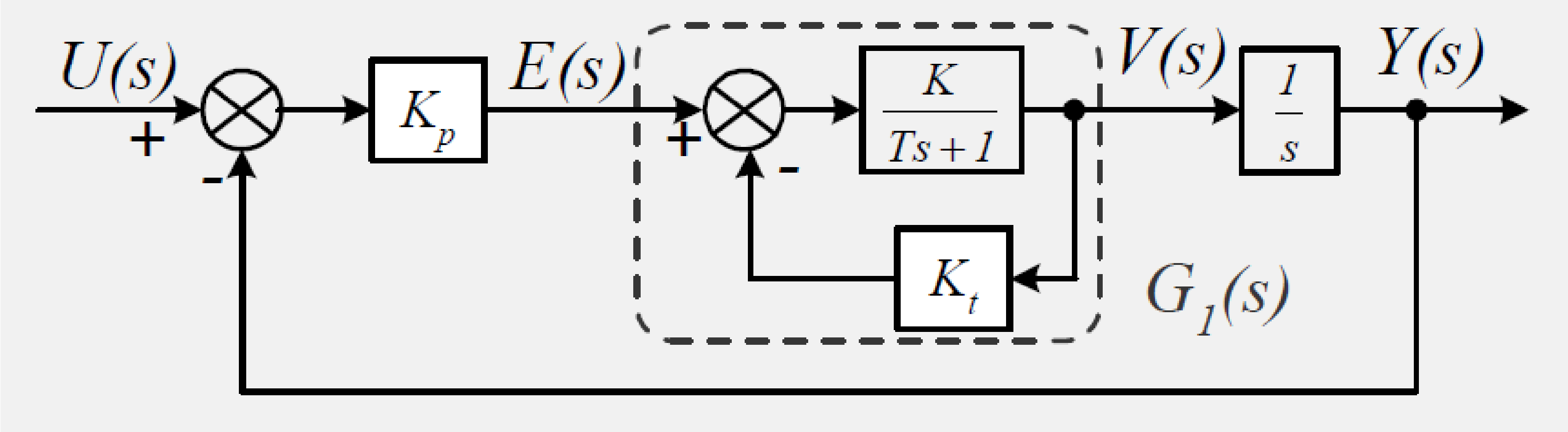
$$2\zeta\omega_n = \frac{1}{T} \quad \Rightarrow \quad \zeta = \frac{1}{2\omega_n T} \approx 0.224$$

$$PO = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \approx 48.6\%$$

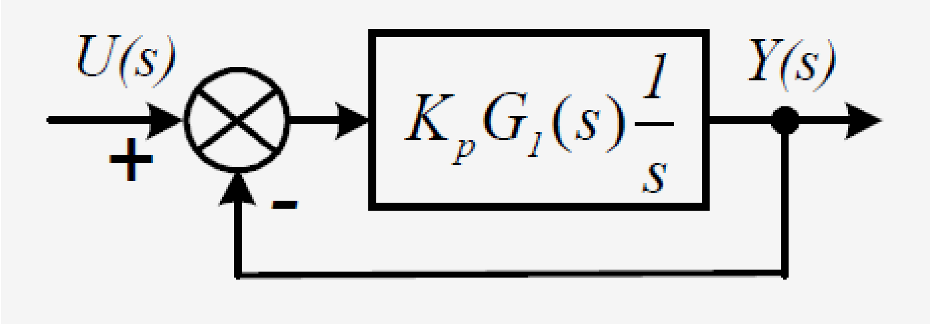
2% settling time: $t_s = \frac{4}{\zeta\omega_n} = 4\,\text{sec}$

Improving 2nd order system performance

- Amplifier gain (K_p) and velocity feedback gain (K_t)



- Find K_p and K_t to achieve:
  - $PO = 15\%$
  - 2% settling time: $t_s = 0.1s$



Improving 2nd order system performance: solution

Firstly, find ζ and ω_n:

$$\zeta = \frac{|\ln(PO)|}{\sqrt{\pi^2 + (\ln(PO))^2}} = \frac{|\ln(0.15)|}{\sqrt{\pi^2 + (\ln(0.15))^2}} \approx 0.52$$

$$t_s = \frac{4}{\zeta\omega_n} = 0.1 \quad \Rightarrow \quad \omega_n = \frac{4}{t_s\zeta} = 76.9$$

– Required transfer function to achieve performance requirements is:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{5913.6}{s^2 + 79.98s + 5913.6}$$

Improving 2nd order system performance: solution

Secondly, find the total system transfer function:

$$G_1(s)=\frac{V(s)}{E(s)}=\frac{\dfrac{K}{1+Ts}}{1+\dfrac{K}{1+Ts}K_t}=\frac{K}{Ts+1+KK_t}$$

$$G(s)=\frac{K_pG_1(s)\dfrac{1}{s}}{1+K_pG_1(s)\dfrac{1}{s}}=\frac{KK_p}{Ts^2+(1+KK_t)s+KK_p}$$

$$=\frac{\left(KK_p/T\right)}{s^2+\left(1+KK_t/T\right)s+\left(KK_p/T\right)}$$

Improving 2nd order system performance: solution

Finally, find K_p and K_t

– Comparing TF with standard 2nd order TF

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{KK_p}{T} = \omega_n^2 \quad \Rightarrow \quad K_p = \frac{\omega_n^2 T}{K} \approx 295.68$$

$$\frac{1 + KK_t}{T} = 2\zeta\omega_n \quad \Rightarrow \quad K_t = \frac{2\zeta\omega_n T - 1}{K} \approx 3.90$$