Multiplication

Rectangular

$z = x + iy$

$\bar{z} = x - iy$ conjugate

$z\bar{z} = (x + iy)(x - iy)$

$= x^2 - \cancel{x(iy)} + \cancel{iy(x)} - iy\,iy$

$= x^2 - i^2 y^2 = x^2 + y^2$

$\boxed{i = \sqrt{-1} \Rightarrow i^2 = -1}$

Multiplication

Polar form

$z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$
$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Multiply the cojugate in polar:

$$z = re^{i\theta}$$
$$\bar{z} = re^{-i\theta} \quad \text{cojugate}$$
$$z\bar{z} = re^{i\theta} re^{-i\theta} = r^2 e^{i(\theta - \theta)} = r^2$$

Division

In dividing complex numbers, we don't want any complex number in the denominater (remove i from the denominator)

Example

$$\frac{1}{i} = ?$$

$$\frac{1}{i}\frac{i}{i} = \frac{i}{i^2} = -i$$

$i = x + iy$, $x = 0$, $y = 1$

$10 = x + iy$, $x = 10$, $y = 0$

Example

$$\frac{1}{2-3i} = ?$$

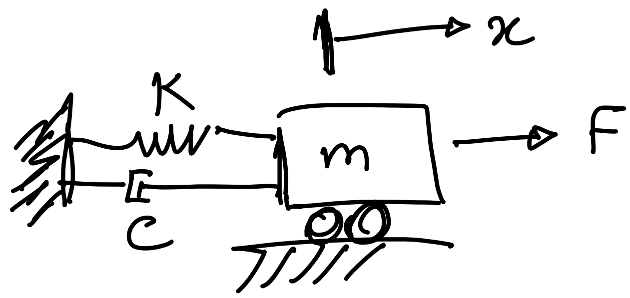
use the conjugate to remove "i" from the denominator:

$$\frac{1}{2-3i}\frac{2+3i}{2+3i} = \frac{2+3i}{4+9} = \frac{2+3i}{13}$$

$$\boxed{(2-3i)(2+3i) = 4-9i^2-6i+6i = 4+9}$$

Example

Find the amplitude of oscilation in the mass-spring example below.



F is the input force

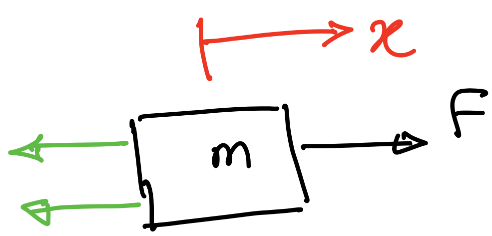
x is the displacement of the mass

K : stiffness of the spring

c : Damping constant of the damper

Free-body-diagram (F.B.D)

Force of the spring: Kx

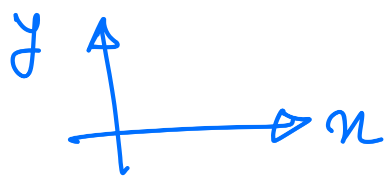


$c\dot{x}$

$$\dot{x} = \frac{dx}{dt}$$

Velocity

Force of the damper



Equations of motion:

$$\xrightarrow{+} \sum \vec{F} = m\vec{a}$$

$$F - c\dot{x} - kx = m\ddot{x}$$

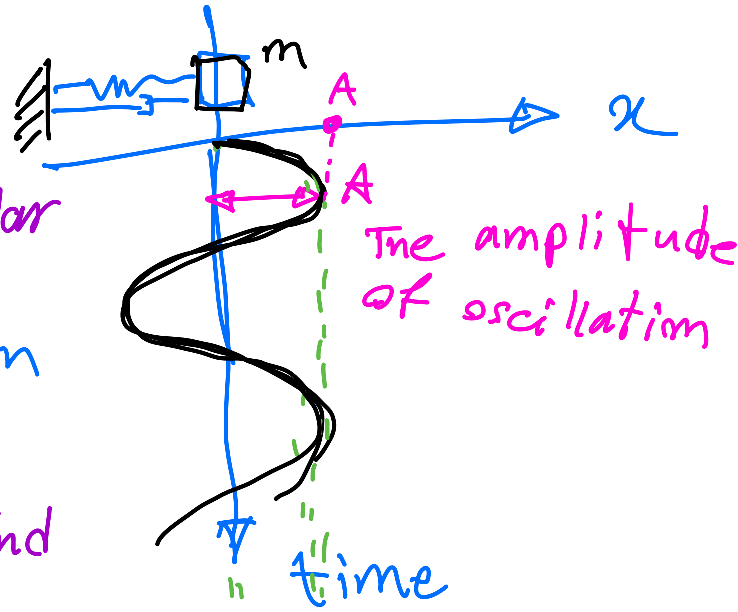
$$\boxed{\ddot{x} = \frac{d^2x}{dt^2} = a \quad \text{acceleration}}$$

Let's assume that the solution for displacement is in the following form:

$$x = Ae^{i\omega t}$$

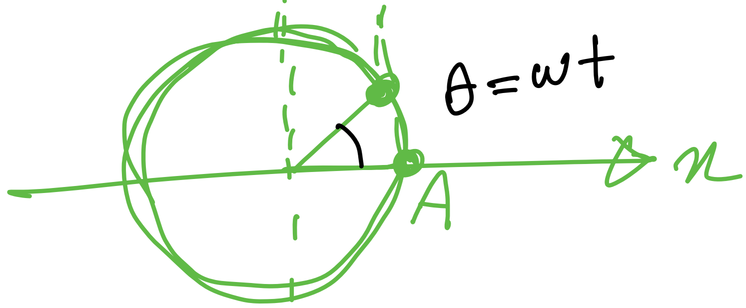
ω is the angular frequency

Amplitude of oscillation



Frequency (Hz) = $\frac{\text{cycle}}{\text{second}}$

ω (angular freq. $\frac{rad}{s}$) = $2\pi f$



θ : Angular displacement

ω = angular velocity

t : time

Displacement = velocity x time

$$\theta = \omega t$$

$$x = Ae^{i\omega t} \longrightarrow \dot{x} = Ai\omega e^{i\omega t}$$

$$\longrightarrow \ddot{x} = A(i\omega)^2 e^{i\omega t}$$

$$= -A\omega^2 e^{i\omega t}$$

$$F - c\dot{x} - kx = m\ddot{x}$$

$$F - c(Ai\omega e^{i\omega t}) - k\,Ae^{i\omega t} = m(-A\omega^2 e^{i\omega t})$$

The response of the mass to the force is a function of the force

If the force is applied with the frequency of ω.

It is expected that the displacement response is a function of the forced excitation with the same frequency.

$$F = Fe^{i\omega t} \Longrightarrow x = Ae^{i\omega t}$$

Force excitation Frequency

$$F - c(Aiwe^{iwt}) - K\,Ae^{iwt} = m(-Aw^2 e^{iwt})$$

$$F = Fe^{iwt}$$

substitute and rearrang:

$$mAw^2 \cancel{e^{iwt}} - icAw\cancel{e^{iwt}} - KA\cancel{e^{iwt}} = -F\cancel{e^{iwt}}$$

$$mAw^2 - icwA - KA = -F$$

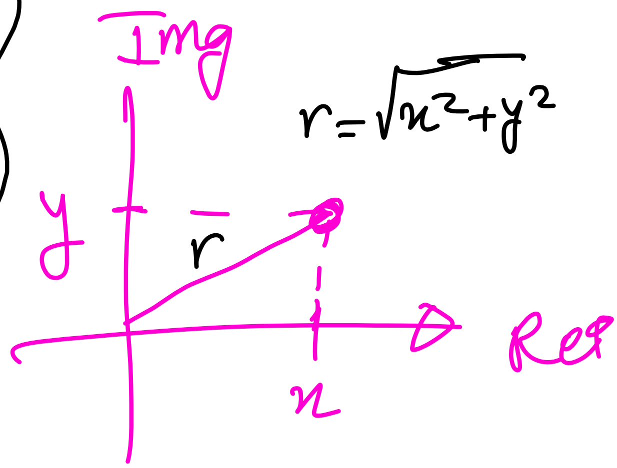
$$A = ?$$

$$A = \frac{-F}{mw^2 - icw - K}$$

$$A = \frac{-F}{\underbrace{(mw^2 - K)}_{\text{Real}} - \underbrace{icw}_{\text{Imag}}}$$

$$A = \frac{-F}{(mw^2-K) - icw} \quad \frac{(mw^2-K)+icw}{(mw^2-K)+icw}$$

$$A = \frac{-F\left[(mw^2-K) - icw\right]}{(mw^2-K)^2 - (cw)^2}$$



$r = \sqrt{x^2+y^2}$

$x = mw^2 + k$

$y = cw$

$$|A| = \frac{F\sqrt{(mw^2-K)^2 + (cw)^2}}{(mw^2-K)^2 + (cw)^2}$$

Amplitude of the oscillation