Complex numbers:

Motivation: $x^2 = 1 \Longrightarrow x = \pm 1$

$x^2 = -1 \Longrightarrow x = ?$

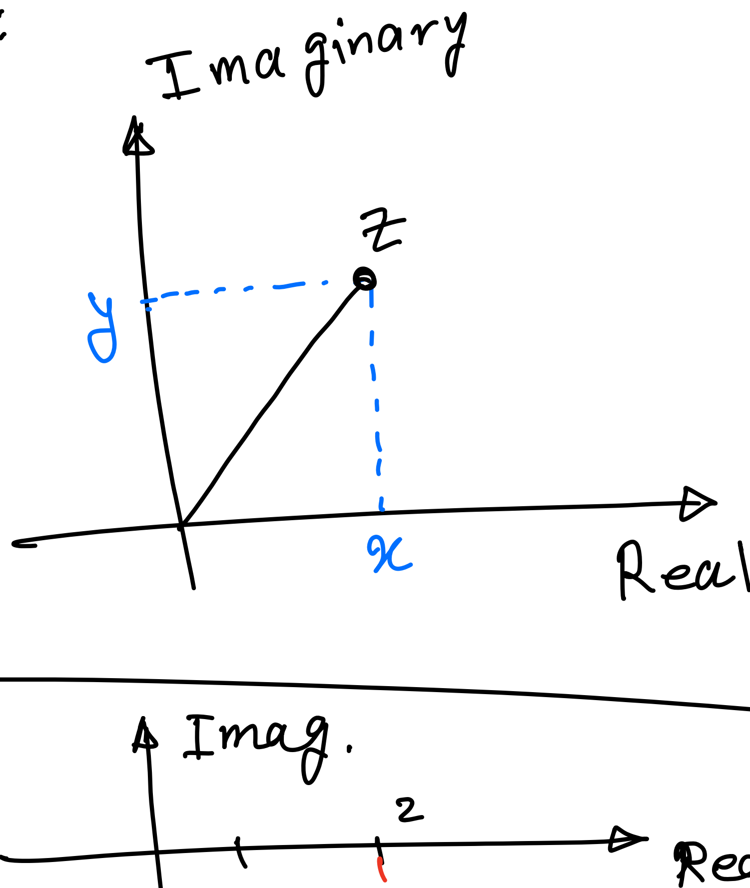
$x = \pm\sqrt{-1}$

we can not calculate $\sqrt{-1}$, then

we use $\sqrt{-1} = i \Longrightarrow i^2 = -1$

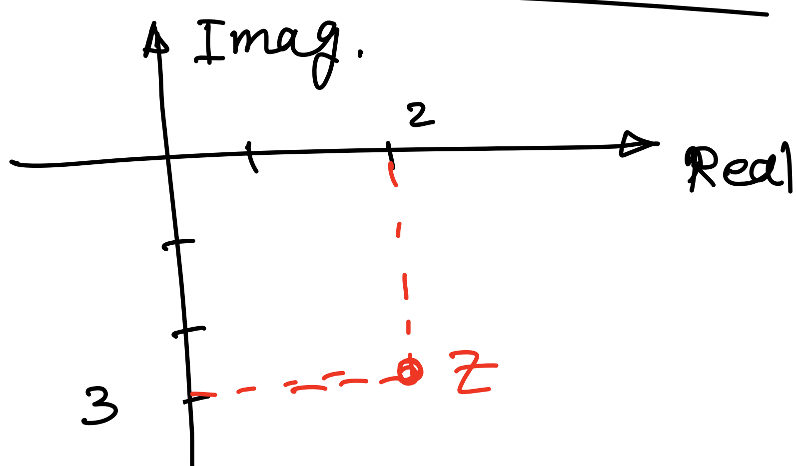
complex numer:

$$z = \underbrace{x}_{\text{Real}} + i\,\underbrace{y}_{\text{Imaginary}}$$



Example

$z = 2 - 3i$

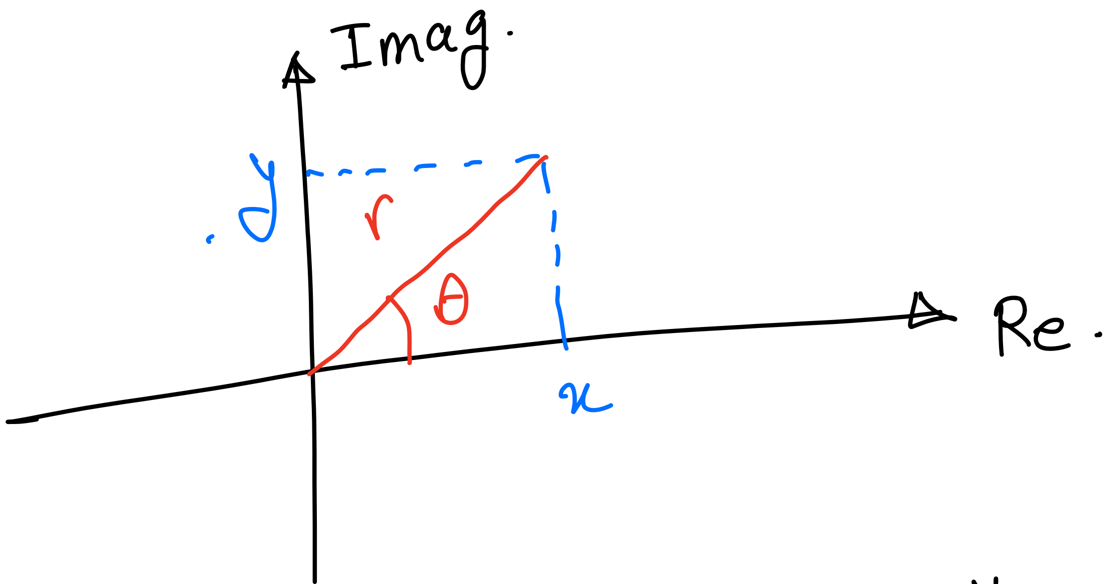


For $z = x + iy$

$x = \text{Real}(z) \longrightarrow \text{Re}(z)$

$y = \text{Imaginary}(z) \longrightarrow \text{Im}(z)$

polar representation of complex numbers:



$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1} \dfrac{y}{x}$

For $z = x + iy$, the polar representation is as follows:

$x = r\cos\theta$ ①

$y = r\sin\theta$ ②

$z = x + iy \xrightarrow{\text{In polar notation}} r\angle\theta \xrightarrow{\text{Alternatively}} re^{i\theta}$

substitute ① and ② into $z = x + iy$:

$z = r\cos\theta + i r\sin\theta$

(It can be shown $\cos\theta + i\sin\theta = e^{i\theta}$)

This is called the Euler's formula

Important : θ must be in radians

Example

$e^{2i} = \cos 2 + i\sin 2$

$e \sim 2.7\ldots$ Euler's number

Conjugate of a complex number:

$$z = x + iy$$

The complex conjugate of z is:

$$\bar{z} = x - iy$$

$$Re(z) = Re(\bar{z}) = x$$

$$Img(z) = -Img(\bar{z}) = y$$

Example

Find the conjugate of

$$z = 2 - 3i$$

$$\bar{z} = 2 + 3i$$

Example

Find $z = 5e^{i\frac{\pi}{4}}$ in rectangular form. The reetangular form is

$x+iy$.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$z = 5e^{i\pi/4} = \underbrace{5\cos\frac{\pi}{4}}_{x=\mathrm{Re}(z)} + i\,\underbrace{5\sin\frac{\pi}{4}}_{y=\mathrm{Imag}(z)}$$

Example

$z = 2-3i$

Express z in polar form of $re^{i\theta}$

$$z = x+iy \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$$

$z = 2-3i \implies x=2$ and $y=-3$

$r = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\tan\theta = \frac{-3}{2} \Longrightarrow \theta = -0.98 \text{ rad}$

The polar form:

$$z = \sqrt{13}\, e^{-i(0.98)}$$

check the answer by using

$z = re^{i\theta} = r\cos\theta + ir\sin\theta$

$z = \sqrt{13}\left[\cos(-0.98) + i\sin(-0.98)\right]$

$= 2 - 3i$ ✓ correct

Operations on complex numbers

1. Adding / subtracting

$z_1 = 3 + 2i$

$z_2 = -4 - 7i$

$$z_1 + z_2 = \left[Re(z_1) + Re(z_2) \right] + i \left[Imag(z_1) + Imag(z_2) \right]$$

$$z_1 + z_2 = -1 - 5i$$

we can not add in polar form.

$$z_1 = r_1 e^{i\theta_1}$$

$$z_2 = r_2 e^{i\theta_2}$$

First convert them from polar to rectangular and then add/subtract.

Multiplication

$$z_1 = 3 + 2i$$

$$z_2 = -4 - 7i$$

$$z_1 z_2 = ?$$

$$z_1 z_2 = (3+2i)(-4-7i)$$

$$= -12 - 21i - 8i - 14i^2$$

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$

$+14$

$$z_1 z_2 = 2 - 29i$$