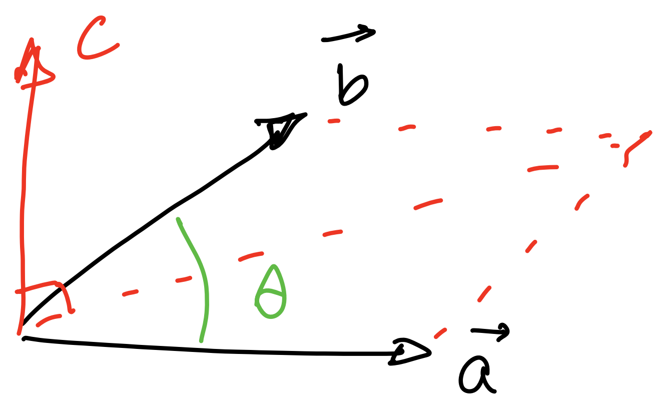
Cross product

The cross product of $\vec{a}$ and $\vec{b}$ is given by:

$$\vec{a} \times \vec{b} = \vec{c}$$

$\vec{c}$ is the cross product of $\vec{a}$ and $\vec{b}$

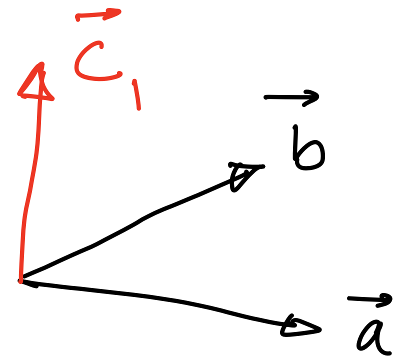


$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\,\sin\theta$$

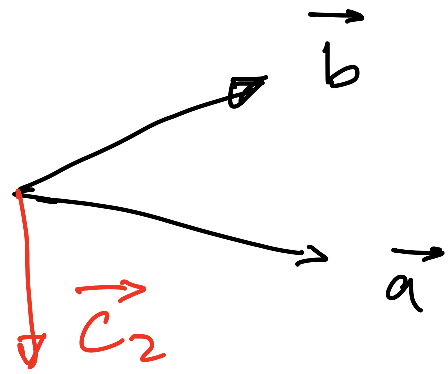
θ is the angle between $\vec{a}$ and $\vec{b}$

The direction of $\vec{c}$ is given by the right hand rule.

$\vec{a} \times \vec{b} = \vec{c}_1$

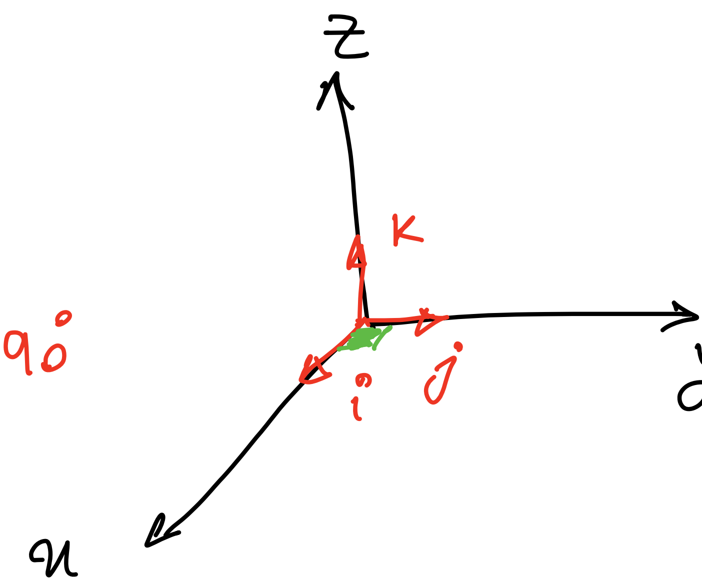


$\vec{b} \times \vec{a} = \vec{c}_2$



$\vec{c}_1 = -\vec{c}_2$

Therefor, the order of the cross product is important.



$\hat{i} \times \hat{j} = \hat{k}$

$\hat{i} \times \hat{j} = |\hat{i}|\,|\hat{j}| \sin 90^\circ$

$= 1 \cdot 1 \cdot 1$

$\hat{i} \times \hat{i} = |\hat{i}|\,|\hat{i}| \sin(0)$

$= 1 \cdot 1 \cdot 0 = 0$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}$$

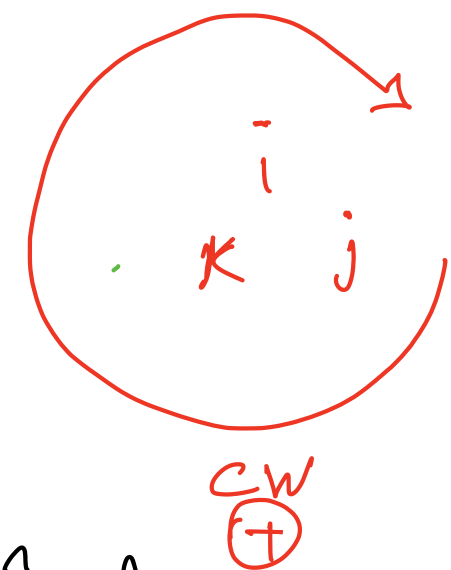
$$\hat{k} \times \hat{j} = -\hat{i}$$

Example

$$\vec{a} = 2\hat{i} - 3\hat{k}$$

$$\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a} \times \vec{b} = ?$$



$$\vec{a} \times \vec{b} = (2\hat{i} - 3\hat{k}) \times (3\hat{i} - \hat{j} + \hat{k})$$

$$= 6\,\underbrace{\hat{i} \times \hat{i}}_{0} - 2\,\underbrace{\hat{i} \times \hat{j}}_{\hat{k}} + 2\,\underbrace{\hat{i} \times \hat{k}}_{-\hat{j}}$$

$$- 9\,\underbrace{\hat{k} \times \hat{i}}_{\hat{j}} + 3\,\underbrace{\hat{k} \times \hat{j}}_{-\hat{i}} - 3\,\underbrace{\hat{k} \times \hat{k}}_{0}$$

$$= -2\hat{k} - 2\hat{j} - 9\hat{j} - 3\hat{i}$$

$$= -3\hat{i} - 11\hat{j} - 2\hat{k}$$

- Now use the determinant method:

$$\vec{a} = 2\hat{i} - 3\hat{k}$$

$$\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a} \times \vec{b} = ?$$

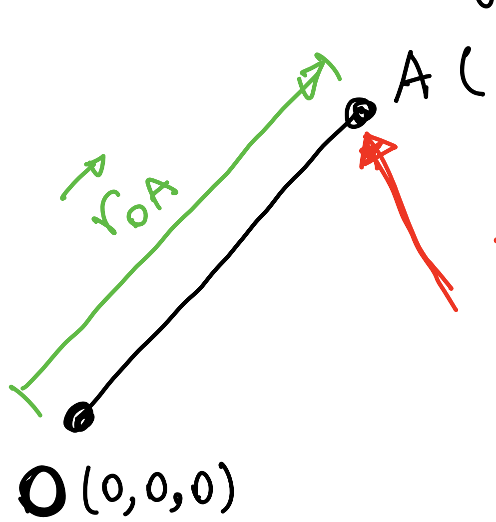
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 3 & -1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 3 & -1 & 1 \end{vmatrix} = \hat{i}\left[(0)(1) - (-3)(-1)\right] - \hat{j}\left[(2)(1) - (-3)(3)\right] + \hat{k}\left[(2)(-1) - (0)(3)\right]$$

$$= -3\hat{i} - 11\hat{j} - 2\hat{k}$$

Example of an application:

Calculating moments:



Calculate the moment of force $\vec{F}$ applied to the rod OA about point O.

$$\vec{M}_O = \vec{r}_{OA} \times \vec{F}$$

$$\vec{M}_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ -2 & 0 & 4 \end{vmatrix} = \hat{i}\,(12) - \hat{j}\,(8-(-10)) + \hat{k}\,(-(3)(-2))$$

$$= 12\hat{i} - 18\hat{j} + 6\hat{k}$$

Example

The force of $F = 30N$ acts on the bracket.

$\alpha = 60^\circ \quad \beta = 60^\circ \quad \gamma = 45^\circ$

Find the moment of $\vec{F}$ about the a-a axis.

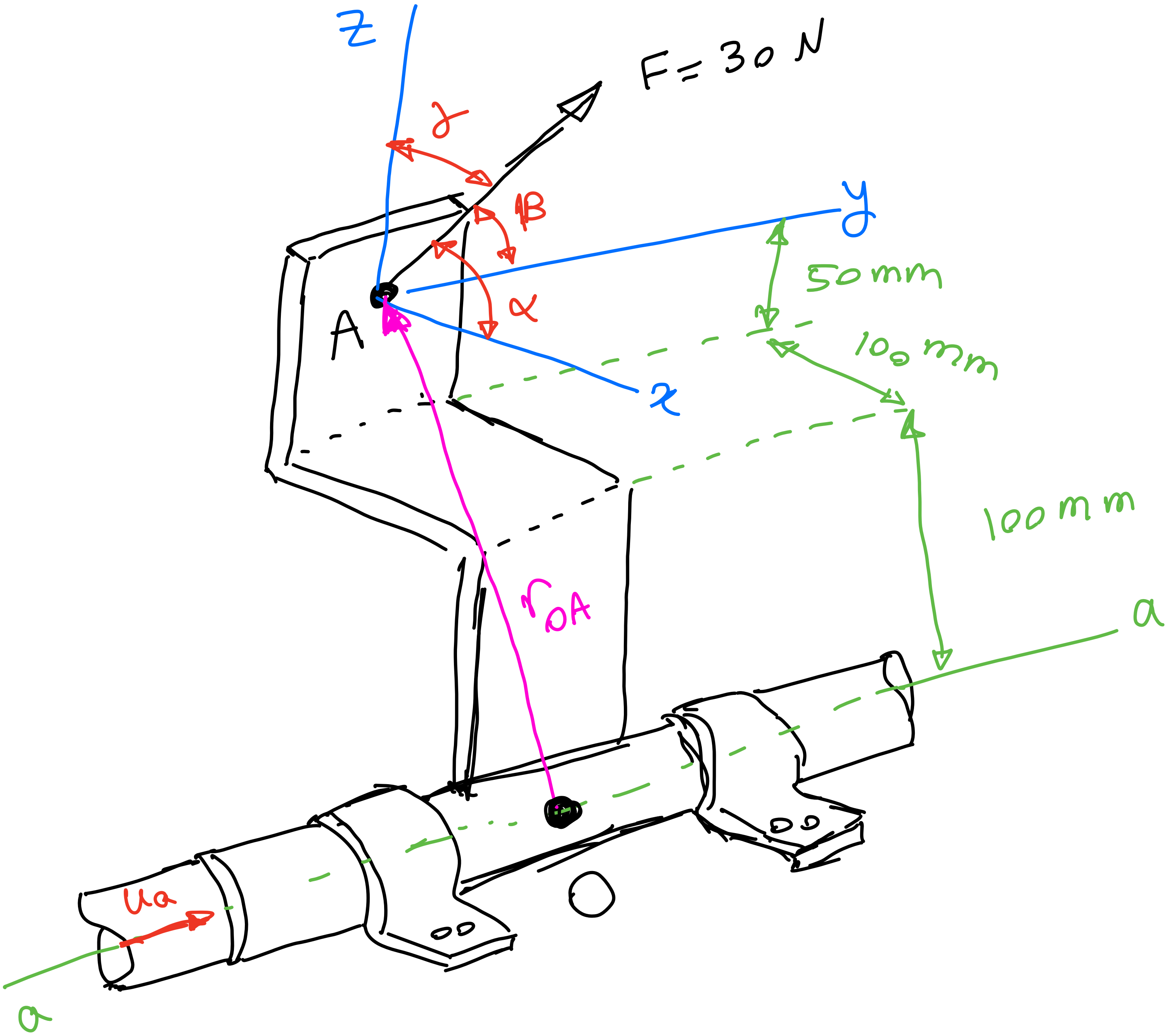
- Find $\vec{u}_a$: $\vec{u}_a = \hat{j}$
- Find $\vec{r}_{oA}$:

$$\vec{r}_{oA} = (x_A - x_o)\hat{i} + (y_A - y_o)\hat{j} + (z_A - z_o)\hat{k}$$

$$= (0 - 100)\hat{i} + (0 - 0)\hat{j} + (0 - (-150))\hat{k}$$

$$= -100\hat{i} + 0\hat{j} + 150\hat{k}$$

or in meters: $\vec{r}_{OA} = -0.1\,\hat{i} + 0.15\,\hat{k}$



– Find $\vec{F}$:

$$\vec{F} = 30\text{ N}\left(\cos 60^\circ\,\hat{i} + \cos 60^\circ\,\hat{j} + \cos 45^\circ\,\hat{k}\right)$$

$$\vec{F} = \left(15\,\hat{i} + 15\,\hat{j} + 21.21\,\hat{k}\right)\text{ N}$$

- Find $\vec{M}_o$:

$$\vec{M}_o = \vec{r}_{OA} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -0.1 & 0 & 0.15 \\ 15 & 15 & 21.21 \end{vmatrix}$$

$$\vec{M}_o = \hat{i}\left[(0)(21.21) - (0.15)(15)\right]$$
$$- \hat{j}\left[(-0.1)(21.21) - (0.15)(15)\right]$$
$$+ \hat{k}\left[(-0.1)(15) - (0)(15)\right]$$
$$= -2.25\,\hat{i} + 4.37\,\hat{j} - 1.5\,\hat{k}$$

$$\vec{u}_a \cdot \vec{M}_a = (\hat{j}) \cdot (-2.25\hat{i} + 4.37\hat{j} - 1.5\hat{k})$$

$$= 4.37$$

The magnitude of the component of the moment along a-a axis

To find this component in a vector form, you can multiply the magnitude by the directio of a-a axis, as below:

$$\vec{M}_{a\text{-}a\ axis} = 4.37\ \vec{u}_a = 4.37\ \hat{j}$$