An application of the weighted discrete Hardy inequality

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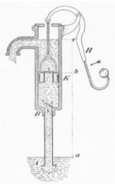

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Theorem (Hardy 1925)

Given $p > 1$, the discrete Hardy inequality claims

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p,$$

for all non-negative sequence $\{a_n\}_{n\geq 1}$.

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  discrete (using series) and continuous (using integrals).

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In this project, we consider the following weighted discrete version:

$$\sum_{n=1}^{\infty} u_n \left(\sum_{k=1}^{n} a_k \right)^p \leq C^p \sum_{n=1}^{\infty} u_n \, a_n^p, \tag{1}$$

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where q is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. In addition,

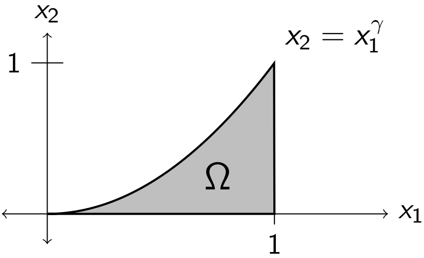
$$\mathbb{A} \leq C \leq 4\mathbb{A}.$$

An application: Solvability of $\operatorname{div} \mathbf{u} = f$ on $\Omega \subset \mathbb{R}^2$

Let us consider the following irregular domain in $\mathbb{R}^2$

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, \text{ and } 0 < x_2 < x_1^\gamma\}$$

where $\gamma > 1$.



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is there a vector field $\mathbf{u} : \Omega \to \mathbb{R}^2$ such that

$$\frac{\partial u_1}{\partial x_1}(x) + \frac{\partial u_2}{\partial x_2}(x) = f(x) \qquad x \text{ in } \Omega$$

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$$\int_\Omega \left| \frac{\partial u_j(x)}{\partial x_i} \right|^2 \mathrm{d}x \le C^2 \int_\Omega |f(x)|^2 \, \mathrm{d}x? \qquad (✿)$$

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For what weights $\omega(x_1)$?

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In addition, we estimate the constant C in (❀) in terms of $\mathbb{A}$.

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Thus, we define

$$r := 2^{-(\gamma+1+2\beta)} < 1.$$

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$$\begin{aligned}
\mathbb{A} &= \sup_{k\geq 1} \left(\sum_{n=k}^{\infty} r^n\right)^{1/2} \left(\sum_{n=1}^{k} r^{-n}\right)^{1/2} \\
&= \sup_{k\geq 1} \left(\frac{r^k}{1-r}\right)^{1/2} \left(\frac{(r^{-1})^{k+1} - r^{-1}}{r^{-1}-1}\right)^{1/2} \\
&< \sup_{k\geq 1} \left(\frac{r^{-1}}{(1-r)(r^{-1}-1)}\right)^{1/2} r^{k/2} r^{-k/2} \\
&= \frac{1}{(1-r)} = \frac{1}{1-2^{-2\left(\beta+\frac{\gamma+1}{2}\right)}} < \infty
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\end{aligned}
$$

which implies, by the characterization, that for any $\{a_n\}_{n\geq 1}$

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\sum_{n=1}^{\infty} u_n \left(\sum_{k=1}^{n} a_k \right)^2 \leq C^2 \sum_{n=1}^{\infty} u_n a_n^2
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Corollary

Let $\beta > \frac{-\gamma-1}{2}$. Then, there exists a solution $\mathbf{u}$ of $\operatorname{div} \mathbf{u} = f$ which satisfies that $\mathbf{u}(x) = 0$ on $\partial\Omega$, and

$$\int_\Omega \left| \frac{\partial u_j(x)}{\partial x_i} \right|^2 x_1^{2(\gamma-1)} x_1^{-2\beta} \, \mathrm{d}x \le C^2 \int_\Omega |f(x)|^2 x_1^{-2\beta} \, \mathrm{d}x, \qquad (\text{❀})$$

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where

$$C^2 \le M \frac{2^{16\beta}}{\left(1 - 2^{-2\left(\beta + \frac{\gamma+1}{2}\right)}\right)^2}$$

for M *independent of* β.

Thank you for your attention!