An application of a weighted discrete Hardy inequality on trees to Friedrichs' inequality on a fractal domain

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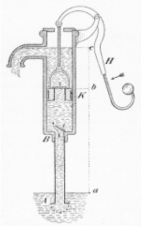

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Friedrichs Inequality

For any Lipschitz domain Ω in $\mathbb{R}^2$ there exists a positive constant C such that any holomorphic function $w(z) = f(z) + ig(z)$ satisfies that

$$\left(\int_\Omega |f(x,y)|^2 \,\mathrm{d}x\mathrm{d}y\right)^{1/2} \leq C \left(\int_\Omega |g(x,y)|^2 \,\mathrm{d}x\mathrm{d}y\right)^{1/2},$$

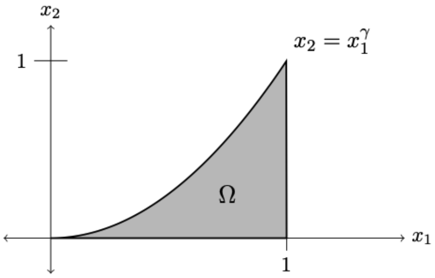
provided $\int_\Omega f = 0$ and assuming $z = x + iy$.

Cauchy-Riemann Equations

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

Friedrichs Inequality

Friedrichs is valid on Lipschitz domains, like rectangles, but not on the cuspidal domain.



Friedrichs Inequality Example

Let $w(z) = z = x + iy$, $\Omega = [-3, 3] \times [-2, 2]$.

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- $w(z)$ holomorphic
- $\int_\Omega f = 0$

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$$\left(\int_\Omega |g(x,y)|^2 \,\mathrm{d}x\mathrm{d}y\right)^{1/2} = \left(\int_\Omega y^2 \,\mathrm{d}x\mathrm{d}y\right)^{1/2} = 32^{1/2} = 4\sqrt{2}$$

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$$\left(\int_\Omega |f|^2 \, \mathrm{d}x\mathrm{d}y \right)^{1/2} \leq \frac{3}{2} \cdot \left(\int_\Omega |g|^2 \, \mathrm{d}x\mathrm{d}y \right)^{1/2}$$

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Importance of the constant

In general, $C \approx \frac{a}{b}$ where a and b are side lengths of the rectangular domain.

Friedrichs Inequality

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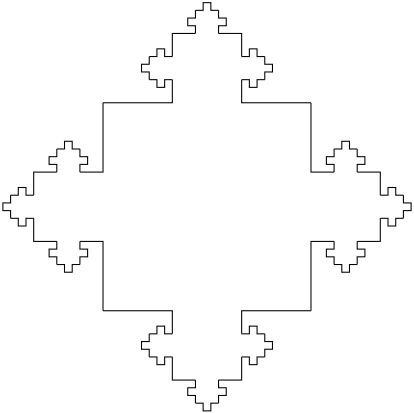
provided $\int_\Omega f = 0$ and assuming $z = x + iy$.

Question

- If we let Ω be a regular domain, is it possible to replace $\mathrm{d}x\mathrm{d}y$ with $d(z)^\beta \,\mathrm{d}x\mathrm{d}y$?
- What β can we have?
- Is β related to the geometry of Ω?

T-square type Fractal



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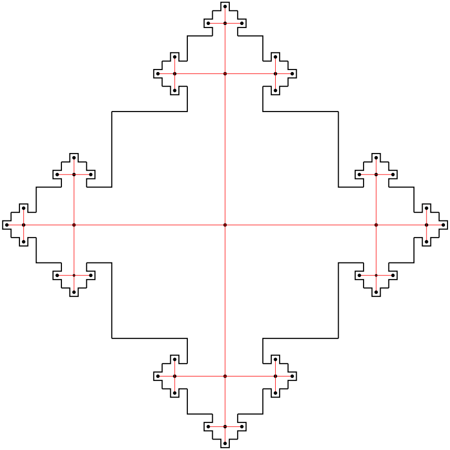
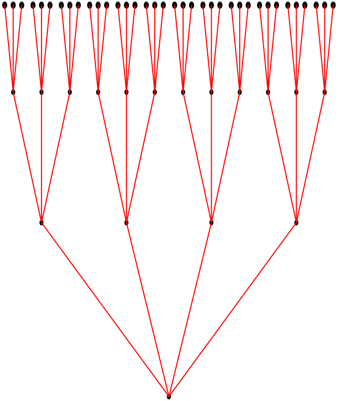


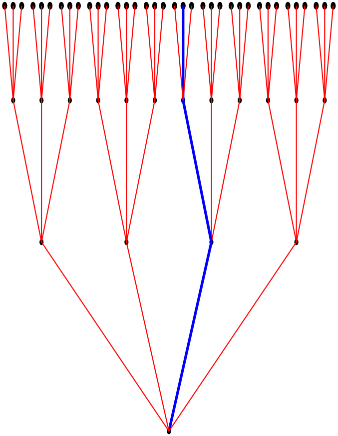


The discrete Hardy inequality (on Trees)

The discrete Hardy inequality states

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2 \leq C\sum_{n=1}^{\infty}a_n^2,$$

for any non-negative sequence $\{a_n\}_{n\geq 1}$



Box-Counting Dimension

Let $F \subset \mathbb{R}^n$. We denote the least number of sets with diameter at most δ as $N_\delta(F)$ which forms a cover for F. Then the *upper* box counting dimension of F is defined as

$$\overline{\dim}_B F = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta} = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

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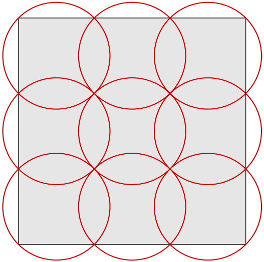

$$\delta = 1/3 : \frac{\log N_\delta(F)}{-\log \delta} \simeq \frac{\log(3^2)}{-\log(1/3)} = 2$$

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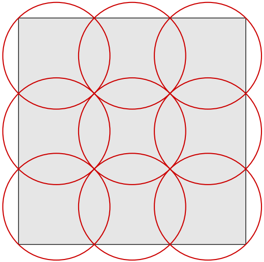

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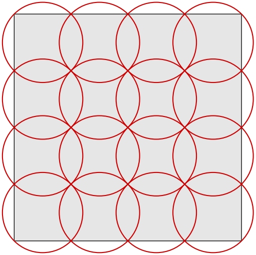

$\delta = 1/4 : \frac{\log N_\delta(F)}{-\log \delta} \simeq \frac{\log(\mathbf{4^2})}{-\log(\mathbf{1}/\mathbf{4})} = 2$

Assouad Dimension

Let $F \subseteq \mathbb{R}^n$. Then we define the Assouad Dimension of F as

$$\dim_A F = \inf \left\{ s : \text{ there exists } C \text{ such that } N_\delta(B(x,R) \cap F) \leq C \left(\frac{R}{\delta}\right)^s \right\}$$

for all $0 < \delta \leq R$ and $x \in F$.

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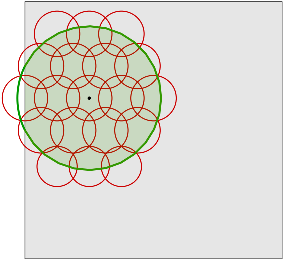

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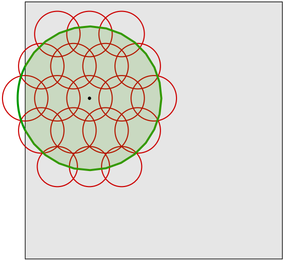

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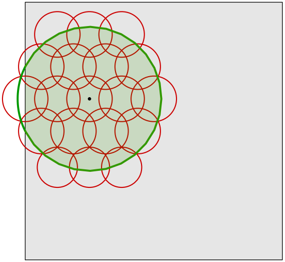

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$$N_{1/8}(x, 1/2) \approx 16 \le C\left(\frac{1/2}{1/8}\right)^1 = C\,(4)^1$$

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In general $\overline{\dim}_B(F) \leq \dim_A(F)$

Results and Further Goals

Theorem. Let $\Omega \subset \mathbb{R}^2$ be the fractal domain T-square, $F \subset \partial\Omega$ and $p\beta > -(2 - \overline{\dim}_B(F))$. There exists a constant C such that any holomorphic function $f(z) + ig(z)$ satisfies that

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provided $\int_\Omega f(z) d_F(z)^{p\beta}\,\mathrm{d}z = 0$, where $d_F(z)$ is the distance to the set F.

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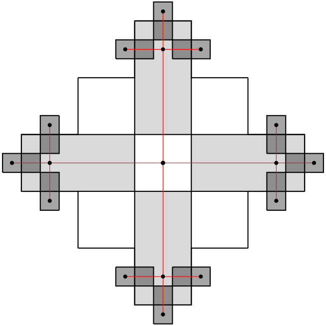
Proof outline. $\Omega \subset \mathbb{R}^2$, $w(z) = f(z) + ig(z)$ holomorphic.



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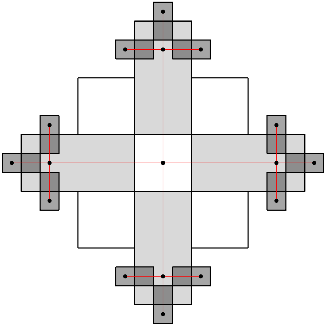

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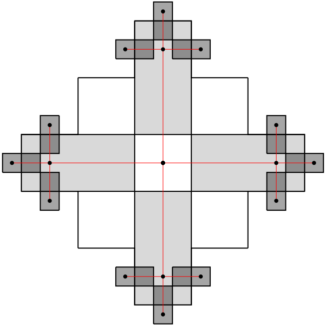

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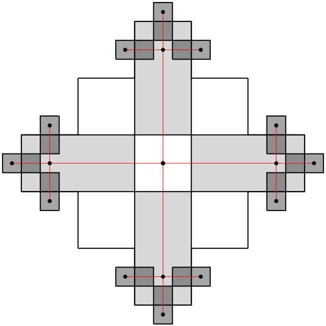

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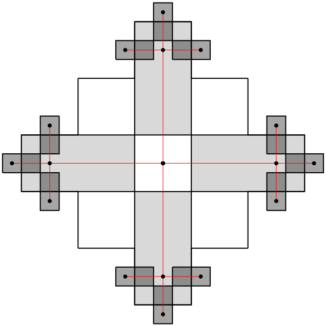

- Use the weighted discrete Hardy inequality to extend the validity of Friedrichs from rectangles to the whole domain Ω

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- $p\beta > -(2 - \overline{\dim}_B(F))$
- $\overline{\dim}_B(F) = \dim_A(F) = 1$