The optimal constant of the Poincaré Inequality on convex domains

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Outline

The classical idea

Theorem (L. E. Payne & H. F. Weinberger '60)

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with diameter d for $n \geq 1$. Then

$$\left(\int_\Omega |u(x) - u_\Omega|^2 \, \mathrm{d}x \right)^{1/2} \leq \frac{d}{\pi} \left(\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \right)^{1/2},$$

where $u \in C^\infty(\Omega)$, and $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u$.

Observe that $\int_\Omega (u(x) - u_\Omega) \, \mathrm{d}x = 0$.

The weighted one-dimensional case

Lemma

Let ρ be a non-negative concave function on the interval $[0, L]$, and let $m \in \mathbb{N}$. Then for any smooth function g satisfying

$$\int_{[0,L]} g(t)\rho^m(t)\,\mathrm{d}t = 0,$$

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we have that

$$\left(\int_{[0,L]} |g(t)|^2 \rho^m(t)\,\mathrm{d}t\right)^{\frac{1}{2}} \leq \frac{L}{\pi}\left(\int_{[0,L]} |g'(t)|^2 \rho^m(t)\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

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Additionally, the constant $\frac{1}{\pi}$ is optimal.

The "slicing" method

Lemma

Let $\Omega \subset \mathbb{R}^n$ *be a convex domain with diameter* d. *Assume that* f *is a smooth function that satisfies* $\int_\Omega f(x)\,\mathrm{d}x = 0$.

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Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d. Assume that f is a smooth function that satisfies $\int_\Omega f(x)\,\mathrm{d}x = 0$. Then for any $\delta > 0$, there are disjoint convex domains $\Omega_i, i = 1, ..., k$ such that

$$\bar{\Omega} = \bigcup_{i=1}^{k} \bar{\Omega}_i, \qquad \int_{\Omega_i} f(x)\,\mathrm{d}x = 0, \quad i = 1, ..., k,$$

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and for each Ω_i, there is a rectangular coordinate system such that

$$\Omega_i \subset \{(x_1, ..., x_n) \in \mathbb{R}^n : 0 \le x_1 \le d_i \text{ and } |x_j| \le \delta, j = 2, ..., n\}.$$

The "slicing" method - proof for planar case ($n = 2$)

Given $\alpha \in [0, 2\pi]$, there is a unique hyperplane H_α with normal

$$v_\alpha = (cos(\alpha), sin(\alpha)),$$

such that H_α divides Ω into two convex sets Ω'_α and Ω''_α with equal area (v_α points in the direction of Ω'_α).

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Define

$$I(\alpha) := \int_{\Omega'_\alpha} f(x)\,\mathrm{d}x.$$

Since $I(\alpha) = -I(\alpha + \pi)$, by continuity there exists an α_0 such that

$$I(\alpha_0) = 0.$$

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We then have that $|\Omega'| = |\Omega''| = \frac{1}{2}|\Omega|$ and

$$\int_{\Omega'} f(x)\,\mathrm{d}x = \int_{\Omega''} f(x)\,\mathrm{d}x = 0 \,\mathrm{at}\, \alpha_0.$$

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Applying this recursively, we divide Ω into a collection of convex subsets Ω_i, for $1 \leq i \leq k$, such that $|\Omega_i|$ is arbitrarily small.

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Then each Ω_i is contained between two parallel hyperplanes of distance at most 2δ.

$$\frac{h^2}{2} \leq \frac{h(d_i)}{2} \leq |\Omega_i| \implies h \leq (2|\Omega_i|)^{\frac{1}{2}} < \delta.$$

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We rotate each Ω_i so that the hyperplanes have normal $v = (0, 1)$, and the desired decomposition is obtained. $\square$

The two-dimensional case

Theorem

Let $\Omega \subset \mathbb{R}^2$ *be a convex domain with diameter d. Then for all smooth f satisfying* $\iint_\Omega f(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0$,

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$$\left(\iint_\Omega |f(x_1, x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \le \frac{d}{\pi}\left(\iint_\Omega |\nabla f(x_1, x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}.$$

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By convexity, $R(x_1) = \rho^{n-1}(x_1)$, where $\rho(x_1)$ is a concave function.

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$$\iint_{\Omega_i} |f(x_1, x_2)|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 = \int_{[0,d_i]} |f(x_1, 0)|^2 R(x_1) \, \mathrm{d}x_1 + E_1(\delta, i)$$

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The two-dimensional case - proof

Some estimates

$$\left| \iint_{\Omega_i} |f(x_1, x_2)|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 - \int_{[0,d_i]} |f(x_1, 0)|^2 R(x_1) \, \mathrm{d}x_1 \right| \leq c_1 |\Omega_i| \delta$$

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$$\left| \iint_{\Omega_i} \left| \frac{\partial f}{\partial x_1}(x_1, x_2) \right|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 - \int_{[0,d_i]} \left| \frac{\partial f}{\partial x_1}(x_1, 0) \right|^2 R(x_1) \, \mathrm{d}x_1 \right| \leq c_3 |\Omega_i| \delta$$

The constants c_1, c_2, and c_3 depend only on f, not on the partition.

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Let

$$g(x_1) := f(x_1, 0) - \frac{1}{|\Omega_i|} \int_{[0,d_i]} f(x_1, 0) R(x_1) \, \mathrm{d}x_1$$

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and by the weighted one-dimensional lemma,

$$\int_{[0,d_i]} |g(x_1)|^2 R(x_1) \, \mathrm{d}x_1 \leq \frac{d_i^2}{\pi^2} \int_{[0,d_i]} |\frac{\partial f}{\partial x_1}(x_1, 0)|^2 R(x_1) \, \mathrm{d}x_1.$$

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By triangle inequality,

$$\int_{[0,d_i]} |f(x_1, 0) R(x_1)|^2 \, \mathrm{d}x_1 \leq \frac{d_i^2}{\pi^2} \int_{[0,d_i]} |\frac{\partial f}{\partial x_1}(x_1, 0)|^2 R(x_1) \, \mathrm{d}x_1 + c_2 f_{\Omega_i} |\Omega_i| \delta.$$

The two-dimensional case - proof

Remark

$$\iint_{\Omega_i} |f(x_1, x_2)|^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \int_{[0,d_i]} |f(x_1, 0)|^2 R(x_1) \,\mathrm{d}x_1 + c_1 |\Omega_i| \delta$$

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and

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So by repeated application of the triangle inequality, we obtain

$$\iint_{\Omega_i} |f(x_1, x_2)|^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \leq \frac{d_i^2}{\pi^2} \iint_{\Omega_i} |\nabla f(x_1, x_2)|^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2$$

$$+ |\Omega_i| \delta (c_1 + c_2 f_{\Omega_i} + c_3 \frac{d_i^2}{\pi^2})$$

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We've now obtained the following inequality over Ω_i:

$$\iint_{\Omega_i} |f(x_1, x_2)|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \leq \frac{d_i^2}{\pi^2} \iint_{\Omega_i} |\nabla f(x_1, x_2)|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

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$$+|\Omega_i|\delta(c_1 + c_2 f_{\Omega_i} + c_3 \frac{d_i^2}{\pi^2})$$

Take the sum of all inequalities on each Ω_i, $1 \leq i \leq k$ to obtain:

$$\iint_{\Omega} |f(x_1, x_2)|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 \leq \frac{d^2}{\pi^2} \iint_{\Omega} |\nabla f(x_1, x_2)|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 + |\Omega|\delta C$$

A comment about $\Omega \subset \mathbb{R}^n$ for $n = 2$ or $n \geq 3$

In the original paper by Payne and Weinberger, it was thought that $R(x)$ is concave, but actually $R(x) = \rho^{n-1}(x)$, with $\rho(x)$ concave.

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This was fixed by M. Bebendorf in 2003, who showed that the original idea works with $R(x) = \rho^m(x)$ for $m \in \mathbb{N}$.

A note on the constant

For bounded convex domains with diameter d, the Poincaré constant is at most $\frac{d}{\pi}$ for L^2 spaces.

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For bounded convex domains with diameter d, the Poincaré constant is at most $\frac{d}{\pi}$ for L^2 spaces.

In certain cases, this constant can be explicitly found. A unit right isosceles triangle, for example, has the constant $\frac{1}{\pi}$ while the diameter is actually $2^{\frac{1}{2}}$.

Thank you for your attention!