Sophisms and Erroneous Resolutions

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Introduction

We aim to develop a resource to help our students get a deeper understanding of the concepts studied in class by using Sophisms and erroneous resolutions, which are based on false logical claims or common misunderstandings. We have prepared several examples involving fallacious and invalid arguments in Pre-Calculus, Calculus, and Introductory Logic. The purpose of these examples is to provide a supplementary resource while also fostering collaboration, argumentation, ability to explain one's thinking and reasoning, self-reflection, and critical thinking.

Background and Goals

Current methods for education in math emphasize correctly worked examples, while disregarding, and sometimes punishing, mistakes. This is known as positive knowledge. Negative knowledge, the opposite, is the use of mistakes to educate. This preference has led to a disdain for mistakes, both from teachers and students. It is common to teach through a mistake and correction process in other classes, such as programming or other language learning classes (Melis, E.). Previous applications of error analysis in classrooms has been promising. It has been shown in a previous experiment that although the short term effects are minimal, long term effects are apparent (Rushton, S. J.).

We wish to develop examples that can be used as a classroom resource for error analysis activities. We intend for these examples to foster self-reflection, critical thinking, an embrace of mistakes and argumentation in students; as well as provoke discussion and collaboration among students. We hope this may help to remedy the positive knowledge bias and aid in development of a well-rounded curriculum.

Results and Conclusion

Although we have not had any formal implementation of our work, we have seen positive feedback when used in an informal setting. Student testimonials report a feeling of deeper understanding, as well as clarity of concepts. Teachers have reported an increase in collaboration and discussion. The informal implementation appears promising, but it is not clear until there are formal tests done.

We have attempted to develop a resource for math educators using Sophisms and Erroneous Resolutions. We have developed three categories of examples. Sophisms, which present logical and conceptual errors, and are meant to encourage collaboration, critical thinking, and self-reflection. Erroneous Resolutions, which present common errors while developing self-reflection and collaboration. Lastly, the Case Study examples present situations without a clear reason for their occurrence, and develops critical thinking, discussion and collaboration. We hope to implement our examples in a formal test setting in the future to obtain clear results.

Sophisms

We develop these examples using errors in logical or conceptual understanding. The purpose of these examples is to promote discussion, foster self-reflection, and encourage critical thinking and collaboration.

Area can be negative...Unlikely!

Let us consider the area below the graph of the positive function $y = \frac{1}{x^2}$ over the interval $[-1, 1]$. Then,

$$A = \int_{-1}^{1} \frac{1}{x^2}\,dx = \frac{-1}{x}\bigg|_{-1}^{1}$$

$$= \frac{-1}{1} - \frac{-1}{-1} = -2.$$

Is this correct?

Discussion

- Proper application of the definition for integration.
  - That is, the hypothesis for integration is not met.
- Emphasize the importance of continuity.
- Relationship between anti-derivatives and their applications.
  - Negative area from a positive function.

Sum of positive numbers can be negative... Really?

Let us consider the sum of positive numbers: $1 + 2 + 4 + 8 + \cdots$.

$$1 + 2 + 4 + 8 + \cdots = \sum_{n=0}^{\infty} 2^n$$

$$= (2-1)\sum_{n=0}^{\infty} 2^n$$

$$= \sum_{n=0}^{\infty} 2 \cdot 2^n - \sum_{n=0}^{\infty} 2^n$$

$$= (2 + 4 + 8 + \ldots) - (1 + 2 + 4 + 8 + \ldots) = -1.$$

Is this possible?

Discussion

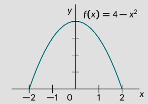
- To manipulate a series this way it must be absolutely convergent.
- This example can be used to develop a similar fallacy, where algebra is done using infinity. Consider the limit below:
  - $\lim_{x\to+\infty} \sqrt{x^2+x} - x = \cdots = 1/2$
  - We can show a mistake by assuming $\infty - \infty = 0$.
- Relationship between operation and elements should not give us this resolution.

Erroneous Resolutions

We develop these examples using common mistakes of techniques or computation. The purpose of these examples is to identify and elaborate on common errors while also fostering self-reflection.

Volume of a solid can be zero!...Impossible?

We wish to compute the volume of the solid generated by revolving the function below about the $y-axis$



We can write the integral for the volume, using cylindrical shells, as:

$$V = 2\pi \int_{-2}^{2} x(4-x^2)\,dx$$

$$= -\pi \int_{u=0}^{u=0} u\,du = 0.$$

We can choose u so that:

$$u = 4 - x^2$$
$$du = -2x\,dx.$$

Could we have made a mistake?

Discussion

- The bounds on our integral are incorrect, we are only revolving over the radius of the figure.
- Students must consider the construction and meaning of formula.
- Relationship between volume and resolution is unreasonable.

Counterexample to uniqueness of limits?

It is clear by evaluation that:

$$\lim_{x\to 1/4} \left(\frac{1}{x}\right) = 4.$$

A student has proposed using squeeze theorem, with the inequality to the right:

$$\frac{1}{2x} < \frac{1}{x} < \frac{1}{\sqrt{x}}$$

The student then showed that:

$$2 = \lim_{x\to 1/4} \left(\frac{1}{2x}\right) \le \lim_{x\to 1/4} \left(\frac{1}{x}\right) \le \lim_{x\to 1/4} \left(\frac{1}{\sqrt{x}}\right) = 2.$$

Is the student's reasoning correct?

Discussion

- The inequalities we choose must hold for all x-values.
  - we must consider non-integer values when checking inequalities.
- Local and Global aspects of the squeeze theorem.
  - inequalities must hold for all x, but locally we must squeeze.
- Consideration for all approaches and study our errors. Don't dismiss approaches.

Case Study

We develop these examples with the purpose of provoking discussion. We hope to further emphasize aspects of collaboration, argumentation, and critical thinking by developing these examples with a Socratic approach.

An Extraneous Solution

We wish to find solutions to the equation

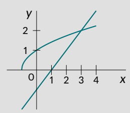
$$\sqrt{x+1} = x - 1.$$

We begin by squaring both sides,

$$(\sqrt{x+1})^2 = (x-1)^2$$
$$x + 1 = x^2 - 2x + 1$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0.$$

Thus, we have solutions $x = 0, 3$.

Now, we will check our solutions by graphing both sides of the equation.



How do we account for the second solution?

Discussion

- What are extraneous solutions? Why do we have to check our solutions? Are we adding solutions? Could there be solutions that are being removed?
- The use of graphs to visualize solutions and understand the problem.
- Not all operations follow in both directions.
  - If we begin with 2 or -2, then we know that $(2)^2 = (-2)^2 = 4$, but if we instead begin with 4, can we tell if we did $(2)^2 = 4$ or $(-2)^2 = 4$.

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