WEIGHTED GENERALIZED KORN INEQUALITIES ON JOHN DOMAINS

FERNANDO LÓPEZ-GARCÍA

ABSTRACT. The goal of this work is to show that the generalized Korn inequality that replaces the symmetric part of the differential matrix in the classical Korn inequality by its trace-free part is valid over John domains and weighted Sobolev spaces. The weights considered are nonnegative powers of the distance to the boundary.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < \infty$. A well-known version of the Korn inequality states that

$$(1.1) \qquad \inf_{\varepsilon(\mathbf{w})=0} \left(\int_\Omega |D\mathbf{v} - D\mathbf{w}|^p \, \mathrm{d}x \right)^{1/p} \le C \left(\int_\Omega |\varepsilon(\mathbf{v})|^p \, \mathrm{d}x \right)^{1/p},$$

for any arbitrary vector field $\mathbf{v}$ in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$, where $n \ge 2$. The constant C depends only on Ω and p, and $\varepsilon(\mathbf{v})$ denotes the symmetric part of the differential matrix $D\mathbf{v}$, namely,

$$\varepsilon(\mathbf{v}) := \frac{D\mathbf{v} + D\mathbf{v}^T}{2}.$$

This inequality is a fundamental result in the analysis of linear elasticity equations where the vector field $\mathbf{v}$ represents the displacement of an elastic body and $\varepsilon(\mathbf{v})$ the linear part of the strain tensor.

The validity of this type of inequalities depend on the geometry of the domain. For example, inequality (1.1) fails on certain domains with cusps (see [1, 2]). The largest known class of domains where this inequality is satisfied is the class of John domains. Examples of John domains are convex domains, Lipschitz domains and the Kock snowflake, a domain with fractal boundary. We refer the reader to [3, 4, 5] where some Korn-type inequalities are studied in this class of domains. The geometric John condition was introduced by Fritz John in [6] and named after him by Martio and Sarvas in [7]. It was recently shown in [8] that certain Korn type inequalities and the John condition are equivalent if the domain satisfies the separation property. This property was introduced by Buckley and Koskela in [9] to exhibit a wide class of domains where the Sobolev-Poincaré inequality and the John condition are equivalent. Examples of domains with the separation property are simply connected plane domains. Reference [9] has also been used to prove the equivalence of the John condition with other inequalities in Sobolev spaces such as the one known as *the inf-sup condition* [3] and an improved version of the fractional Sobolev-Poincaré inequality [10].

In this work we deal with a generalized version of (1.1) known simply as the generalized Korn inequality or the conformal Korn inequality, where the linearized strain vector $\varepsilon(\mathbf{v})$ is replaced by its trace-free part $l(\mathbf{v})$ defined by:

$$l(\mathbf{v}) := \varepsilon(\mathbf{v}) - \frac{\operatorname{div} \mathbf{v}}{n} I,$$

where I in $\mathbb{R}^{n \times n}$ is the identity matrix. More specifically, the main goal of these notes is proving that the inequality

$$(1.2) \qquad \inf_{l(\mathbf{w})=0} \left(\int_\Omega |D\mathbf{v} - D\mathbf{w}|^p \, \mathrm{d}x \right)^{1/p} \leq C \left(\int_\Omega |l(\mathbf{v})|^p \, \mathrm{d}x \right)^{1/p},$$

is valid for any vector field $\mathbf{v}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded John domain and $n \geq 3$.

Different types of the Korn inequality involving the trace-free part of the symmetric gradient have been recently studied for their interest as a mathematical result and for their applications, for instance, to general relativity and Cosserat elasticity. We refer the reader to the following articles and references therein for more details [11, 12, 13, 14]. However, we are especially interested in two Korn-type inequalities published in [15] and [16] where no assumptions on the values of the vector fields over the boundary of the domain are considered. In the first article, Yu. G. Reshetnyak showed the following generalized Korn inequality over star-shaped domains with respect to a ball in $\mathbb{R}^n$, where $n \geq 3$ and $1 < p < \infty$:

$$(1.3) \qquad \|\mathbf{v} - \Pi(\mathbf{v})\|_{W^{1,p}(\Omega)} \leq C\|l(\mathbf{v})\|_{L^p(\Omega)},$$

valid for all $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^n)$. The operator $\Pi : W^{1,p}(\Omega, \mathbb{R}^n) \to \Sigma$ is a projection (i.e. a continuous linear operator such that $\Pi(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \Sigma$), where Σ is the kernel of l and it is endowed with the topology of $L^p(\Omega, \mathbb{R}^n)$. This result was proved by using a certain integral representation of the vector function $\mathbf{v} - \Pi(\mathbf{v})$ in terms of $l(\mathbf{v})$ and then the theory of singular integral operators. Let us recall that the class of star-shaped domains with respect to a ball contains convex domains and it is strictly included in the class of John domains. The study of this inequality was motivated by its connexion with the stability of Liouville's theorem.

The second Korn-type inequality of our interest was published in [16] and says

$$(1.4) \qquad \|\mathbf{v}\|_{W^{1,2}(\Omega)} \leq C\{\|(\mathbf{v})\|_{L^2(\Omega)} + \|l(\mathbf{v})\|_{L^2(\Omega)}\},$$

where $\mathbf{v}$ is an arbitrary vector field in $W^{1,2}(\Omega, \mathbb{R}^n)$ and C depends only on Ω. In this case, Ω is an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$, with $n \geq 3$. This theorem was proved in [16] by using the classical result known as Lions Lemma that claims that any distribution in the space $H^{-1}(\Omega)$ with gradient in $H^{-1}(\Omega)^n$ belongs to $L^2(\Omega)$. A generalized version of this result for distributions in $H^{-2}(\Omega)$ is also required. It is shown in [16] that inequality (1.4) fails on planar domains independently of the geometry of the domain.

The main result of this work states that generalized Korn inequality (1.2) holds also in weighted Sobolev spaces on John domains where the weights are nonnegative powers of the distance to the boundary.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain with $n \geq 3$, $1 < p < \infty$ and $\beta \geq 0$. Then, there exists a constant C such that*

$$(1.5) \qquad \inf_{l(\mathbf{w})=0} \left(\int_\Omega |D\mathbf{v} - D\mathbf{w}|^p \rho^{p\beta} \, \mathrm{d}x \right)^{1/p} \leq C \left(\int_\Omega |l(\mathbf{v})|^p \rho^{p\beta} \, \mathrm{d}x \right)^{1/p}$$

for all vector field $\mathbf{v}$ from Ω to $\mathbb{R}^n$ in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n, \rho^{p\beta})$. The function $\rho(x)$ is the distance from x to the boundary of Ω.

As a Corollary of the main theorem, we prove that (1.3) and (1.4) are also valid on bounded John domains.

Let us recall the definition of John domains. A bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is called a *John domain* with parameter $C > 1$ if there exists a point $x_0 \in \Omega$ such that every $y \in \Omega$ has a rectifiable curve parameterized by arc length $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = y$, $\gamma(l) = x_0$ and

$$\text{dist}(\gamma(t), \partial\Omega) \geq \frac{1}{C} t$$

for all $t \in [0, l]$, where l is the length of γ.

Finally, notice that powers of the distance to the boundary are not in the Muckenhoupt class (A_p class) for arbitrarily large exponents. Thus, approaches that rely on the continuity of the Hardy-Littlewood maximal operator do not apply to the current setting. Instead of the maximal operator we use a Hardy type operator T, which definition (4.1) depends on the geometry of the domain and it is continuous for this type of weights. Moreover, if Ω is a bounded John domain it follows that T is pointwise bounded by the Hardy-Littlewood maximal operator (after multiplying by a constant), then it is also continuous for weights in the Muckenhoupt class. Since this paper is based on a local-to-global argument, extending Theorem 1.1 to arbitrary Muckenhoupt weights would require to prove first the result on cubes. For general knowledge, we refer the reader to [17, 18] where necessary conditions on a compact set F and a real number β are given such that the power distance $d^\beta(x, F)$ becomes a Muckenhoupt weight.

2. NOTATION

Throughout the paper $\Omega \subset \mathbb{R}^n$ denotes a bounded domain where $n \geq 3$ and $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Given $\eta : \Omega \to \mathbb{R}$ a positive measurable function we denote by $L^p(\Omega, \eta)$ the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ equipped with the norm:

$$\|u\|_{L^p(\Omega,\eta)} := \left(\int_\Omega |u(x)|^p \eta(x) \, \mathrm{d}x \right)^{1/p}.$$

Analogously, we define the weighted Sobolev spaces $W^{1,p}(\Omega, \eta)$ as the space of weakly differentiable functions $u : \Omega \to \mathbb{R}$ with the norm:

$$\|u\|_{W^{1,p}(\Omega,\eta)} := \left(\int_\Omega |u(x)|^p \eta(x) \, \mathrm{d}x + \sum_{i=1}^n \int_\Omega \left| \frac{\partial u(x)}{\partial x_i} \right|^p \eta(x) \, \mathrm{d}x \right)^{1/p}.$$

We extend this definition to functions from Ω to $\mathbb{R}^{n \times n}$, and from Ω to $\mathbb{R}^n$, denoted by $L^p(\Omega, \mathbb{R}^{n \times n}, \eta)$, and $W^{1,p}(\Omega, \mathbb{R}^n, \eta)$ respectively.

Given $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and $1 \leq r \leq \infty$ we denote by $\|\tilde{g}\|_r : \mathbb{R}^n \to \mathbb{R}$ the following function

$$\|\tilde{g}\|_r(x) := \left(\sum_{1 \leq i,j \leq n} |\tilde{g}_{ij}(x)|^r \right)^{1/r}$$

if $r \neq \infty$, and

$$\|\tilde{g}\|_\infty(x) := \max_{1 \leq i,j \leq n} |\tilde{g}_{ij}(x)|.$$

Notice that for any $1 \leq r_1, r_2 \leq \infty$ there is a positive constant $C = C(r_1, r_2)$ such that

$$\frac{1}{C}\|\tilde{g}\|_{r_1}(x) \leq \|\tilde{g}\|_{r_2}(x) \leq C\|\tilde{g}\|_{r_1}(x)$$

for all functions $\tilde{g}$ and $x \in \Omega$.

Moreover, $|\tilde{g}| : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is given by $|\tilde{g}|_{ij}(x) = |\tilde{g}_{ij}(x)|$. We say that the function $\tilde{g}$ is integrable (similarly bounded) if each coordinate is integrable (bounded).

For $\tilde{g}, \tilde{f} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ we denote by $\tilde{g}(x) : \tilde{f}(x)$ the product coordinate by coordinate

$$\tilde{g}(x) : \tilde{f}(x) := \sum_{1 \leq i,j \leq n} \tilde{g}_{ij}(x)\tilde{f}_{ij}(x).$$

We say that x belongs to $supp(\tilde{g})$ iff $\tilde{g}(x)$ has at least one coordinate different from zero. We denote with tilde those functions with codomain in $\mathbb{R}^{n \times n}$.

Finally, a *Whitney decomposition* of Ω is a collection $\{Q_t\}_{t \in \Gamma}$ of closed dyadic cubes whose interiors are pairwise disjoint which satisfy

(1) $\Omega = \bigcup_{t \in \Gamma} Q_t$,
(2) $\operatorname{diam}(Q_t) \leq \operatorname{dist}(Q_t, \partial\Omega) \leq 4\operatorname{diam}(Q_t)$,
(3) $\frac{1}{4}\operatorname{diam}(Q_s) \leq \operatorname{diam}(Q_t) \leq 4\operatorname{diam}(Q_s)$, if $Q_s \cap Q_t \neq \emptyset$.

Two different cubes Q_s and Q_t with $Q_s \cap Q_t \neq \emptyset$ are called *neighbors*. Notice that two neighbors may have an intersection with dimension less than $n - 1$. For instance, they could be intersecting each other in a one-point set. We say that Q_s and Q_t are $(n-1)$-neighbors if $Q_s \cap Q_t$ is the $n - 1$ dimensional face of one of them. This kind of covering exists for any proper open set in $\mathbb{R}^n$ (see [19] for details). Moreover, each cube Q_t has less than 12^n neighbors. And, if we fix $0 < \epsilon < \frac{1}{4}$ and define Q_t^* as the cube with the same center as Q_t and side length $(1 + \epsilon)$ times the side length of Q_t, then Q_t^* touches Q_s^* if and only if Q_t and Q_s are neighbors.

Given a Whitney decomposition $\{Q_t\}_{t \in \Gamma}$ of Ω we refer by an *extended Whitney decomposition* of Ω to the collection of open cubes $\{\Omega_t\}_{t \in \Gamma}$ defined by

$$\Omega_t := \frac{9}{8}Q_t^\circ.$$

Observe that this collection of cubes satisfies that

$$\chi_\Omega(x) \leq \sum_{t \in \Gamma} \chi_{\Omega_t}(x) \leq 12^n \chi_\Omega(x)$$

for all $x \in \mathbb{R}^n$.

3. Proof of the main result

This section is devoted to showing Theorem 1.1. The proof follows from a local-to-global argument based on the validity of (1.5), with $\beta = 0$, on cubes and a certain decomposition of functions stated in Lemma 3.4 which is proved in Section 4.

The following result is implied by the validity of (1.3) proved by Reshetnyak [15] over star-shaped domains with respect to a ball.

Corollary 3.1. *Let $Q \subset \mathbb{R}^n$ be an arbitrary cube with sides parallel to the axis, with $n \geq 3$, and $1 < p < \infty$. Then, there exists a positive constant C that depends only on n and p such that*

$$(3.1) \qquad \inf_{l(\mathbf{w})=0} \left(\int_Q |D\mathbf{v} - D\mathbf{w}|^p \, dx \right)^{1/p} \leq C \left(\int_Q |l(\mathbf{v})|^p \, dx \right)^{1/p}$$

for all $\mathbf{v} \in W^{1,p}(Q, \mathbb{R}^n)$.

Proof. Cubes are star-shaped domains with respect to a ball, thus from (1.3) we can conclude that

$$\inf_{l(\mathbf{w})=0} \|D\mathbf{v} - D\mathbf{w}\|_{L^p(Q)} \leq \|\mathbf{v} - \Pi(\mathbf{v})\|_{W^{1,p}(Q)} \leq C\|l(\mathbf{v})\|_{L^p(Q)},$$

where C depends on p, n and Q. It only remains to prove that there is a uniform constant that makes (3.1) valid for any arbitrary cube with sides parallel to the axis. Let Q_0 be the cube $(0,1)^n$ with constant C_{Q_0} in the inequality (3.1). Hence, any other cube with sides parallel to the axis can be obtained by a translation and dilation of Q_0. Now, the inequality only involves first order partial derivatives, thus by making a change of variable, we can extend the validity of this Korn type inequality from Q_0 to any other cube in the statement of corollary with the same constant C_{Q_0}. $\square$

The kernel of the operator l, denoted by Σ in these notes, plays a central role in this local-to-global argument. So, let us recall its characterization which is significantly different if $n = 2$ or $n \geq 3$. In the planar case, Σ is an infinite dimensional space where $\mathbf{w} \in \Sigma$ iff $\mathbf{w}(x,y) = (w_1(x,y), w_2(x,y))$ where w_1 and w_2 are the components of an analytical function $F(x + iy) := w_1(x,y) + iw_2(x,y)$. The fact that the kernel has infinite dimension and the well-known Rellich-Kondrachov Theorem for Sobolev spaces imply the failure of (1.4) for planar domains (see [16]). We have described the planar case for general knowledge, however, in this article, we deal with $n \geq 3$. In that case, when $n \geq 3$, the kernel of l has a finite dimension equal to $\frac{(n+1)(n+2)}{2}$ and a vector field $\mathbf{w} \in \Sigma$ iff

$$(3.2) \qquad \mathbf{w}(x) = a + A(x - y) + \lambda(x - y) + \left\{ \langle b, x - y \rangle (x - y) - \frac{1}{2}|x - y|^2 b \right\},$$

where $A \in \mathbb{R}^{n \times n}$ is skew-symmetric, $a, b \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The vector $y \in \mathbb{R}^n$ is arbitrary but must be fixed to have uniqueness for this representation.

Now, we define the space $\mathcal{V}$ whose elements are the differential matrix of the vector fields in Σ. Namely,

$$\mathcal{V} := \{ \tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^{n \times n} : \tilde{\varphi}(x) = A + \lambda I + \sum_{i=1}^n b_i H_i(x - y)$$

$$\text{with } A \text{ skew-symmetric, } \lambda \in \mathbb{R} \text{ and } b_i \in \mathbb{R} \text{ for all } i\}.$$

The matrix I is the identity and, for $1 \leq i \leq n$, $H_i(x)$ is the functions with values in $\mathbb{R}^{n \times n}$ defined by

$$(3.3) \qquad (H_i)_{jk}(z) = \begin{cases} z_i & \text{if } j = k \\ z_j & \text{if } j \neq k, \ k = i \\ -z_k & \text{if } j \neq k, \ j = i \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq j, k \leq n$. In the particular case when $n = 3$, we have:

$$H_1 = \begin{pmatrix} z_1 & -z_2 & -z_3 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \end{pmatrix} \quad H_2 = \begin{pmatrix} z_2 & z_1 & 0 \\ -z_1 & z_2 & -z_3 \\ 0 & z_3 & z_2 \end{pmatrix} \quad H_3 = \begin{pmatrix} z_3 & 0 & z_1 \\ 0 & z_3 & z_2 \\ -z_1 & -z_2 & z_3 \end{pmatrix}.$$

Observe that the definition of $\mathcal{V}$ does not depend on the vector $y \in \mathbb{R}^n$. By taking a different y we only obtain a different representation of the functions $\tilde{\varphi}(x)$ in $\mathcal{V}$. Thus, let us denote by $m := \frac{n(n-1)}{2} + 1 + n$ the dimension of $\mathcal{V}$. Finally, to prove that $\tilde{\varphi}$ belongs to $\mathcal{V}$ iff $\tilde{\varphi} = D\mathbf{w}$ for some $\mathbf{w} \in \Sigma$ it is sufficient to show that the quadratic part appearing between brackets in (3.2), denoted by $\mathbf{r}$ for simplicity, is $\sum_{i=1}^{n} b_i H_i(x - y)$. Indeed,

$$\frac{\partial \mathbf{r}_j}{\partial x_k} = \begin{cases} \sum_{i=1}^{n} b_i(x_i - y_i) & \text{if } j = k \\ b_k(x_j - y_j) - b_j(x_k - y_k) & \text{if } j \neq k. \end{cases}$$

Thus, $\left(\frac{\partial \mathbf{r}_j}{\partial x_k}\right)_{jk} = \sum_{i=1}^{n} b_i H_i(x - y)$ concluding that

$$\mathcal{V} = D(\Sigma).$$

Now, we define the subspace $\mathcal{W} \subset L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$ by:

$$\mathcal{W} := \{\tilde{g} \in L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta}) : \int \tilde{g} : \tilde{\varphi} = 0 \text{ for all } \tilde{\varphi} \in \mathcal{V}\},$$

where the product ":" is the standard inner product for vectors understanding matrices in $\mathbb{R}^{n \times n}$ as vectors in $\mathbb{R}^{n^2}$. Notice that $\rho^{p\beta}$ belongs to $L^1(\Omega)$ thus

$$L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta}) \subset L^1(\Omega, \mathbb{R}^{n \times n}).$$

Moreover, using that Ω is bounded it follows that $\mathcal{V} \subset L^\infty(\Omega, \mathbb{R}^{n \times n})$. Hence, $\mathcal{W}$ is well-defined.

Lemma 3.2. *The space $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$ can be written as $\mathcal{W} \oplus \rho^{p\beta}\mathcal{V}$. Moreover, for all $\tilde{F} = \tilde{g} + \rho^{p\beta}\tilde{\psi}$ in $\mathcal{W} \oplus \rho^{p\beta}\mathcal{V}$ it follows that $\|\tilde{g}\|_{L^q(\Omega, \rho^{-q\beta})} \leq C_1 \|\tilde{F}\|_{L^q(\Omega, \rho^{-q\beta})}$ where*

$$(3.4) \qquad C_1 = \left(1 + \sum_{j=1}^{m} \|\tilde{\psi}_j\|_{L^p(\Omega, \rho^{p\beta})} \|\tilde{\psi}_j\|_{L^q(\Omega, \rho^{q\beta})}\right).$$

The collection $\{\tilde{\psi}_j\}_{1 \leq j \leq m}$ in the previous identity is an arbitrary orthonormal basis of $\mathcal{V}$ with respect to the inner product

$$\langle \tilde{\psi}, \tilde{\varphi} \rangle_\Omega = \int_\Omega \tilde{\psi}(x) : \tilde{\varphi}(x) \, \rho^{p\beta}(x) \, dx.$$

Proof. Notice that $\rho^{p\beta}\tilde{\psi}$, with $\tilde{\psi} \in \mathcal{V}$, belongs to $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$. Indeed,

$$\|\rho^{p\beta}\tilde{\psi}\|^q_{L^q(\Omega, \rho^{-q\beta})} = \int_\Omega \|\tilde{\psi}\|^q_q \rho^{pq\beta} \omega^{-q\beta} \leq \left(\sup_{x \in \Omega} \|\tilde{\psi}(x)\|^q_q \right) \int_\Omega \rho^{p\beta}(x) \, \mathrm{d}x.$$

Thus, $\rho^{p\beta}\mathcal{V}$ is a subspace of $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$.

The representation follows naturally from the definition of $\mathcal{W}$. Indeed, given $\tilde{F}$ in the space $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$ we take

$$(3.5) \qquad \tilde{\psi}_{\tilde{F}}(x) := \sum_{j=1}^m \alpha_{\tilde{F},j} \tilde{\psi}_j(x),$$

where $\alpha_{\tilde{F},j} := \int_\Omega \tilde{F} : \tilde{\psi}_j$ for any $1 \leq j \leq m$. Thus, $\tilde{F} = \tilde{h}_{\tilde{F}} + \rho^{p\beta}\tilde{\psi}_{\tilde{F}}$ where $\tilde{\psi}_{\tilde{F}} \in \mathcal{V}$ and $\tilde{h}_{\tilde{F}} := \tilde{F} - \rho^{p\beta}\tilde{\psi}_{\tilde{F}} \in \mathcal{W}$. The uniqueness is a simple exercise of linear algebra. Now, to obtain (3.4) notice that the coefficients $\alpha_{\tilde{F},j}$ verify

$$(3.6) \qquad |\alpha_{\tilde{F},j}| \leq \|\tilde{F}\|_{L^q(\Omega, \rho^{-q\beta})} \|\tilde{\psi}_j\|_{L^p(\Omega, \rho^{p\beta})},$$

for all j. Thus, from (3.5) and (3.6) we have

$$\|\tilde{h}_{\tilde{F}}\|_{L^q(\Omega, \rho^{-q\beta})} \leq \left(1 + \sum_{j=1}^m \|\tilde{\psi}_j\|_{L^p(\Omega, \rho^{p\beta})} \|\rho^{p\beta}\tilde{\psi}_j\|_{L^q(\Omega, \rho^{-q\beta})} \right) \|\tilde{F}\|_{L^q(\Omega, \rho^{-q\beta})}$$

$$= \left(1 + \sum_{j=1}^m \|\tilde{\psi}_j\|_{L^p(\Omega, \rho^{p\beta})} \|\tilde{\psi}_j\|_{L^q(\Omega, \rho^{q\beta})} \right) \|\tilde{F}\|_{L^q(\Omega, \rho^{-q\beta})}.$$

$\square$

Given a decomposition of extended Whitney cubes $\{\Omega_t\}_{t \in \Gamma}$ of Ω, we define the subspace $\mathcal{W}_0 \subset \mathcal{W}$ by:

$$\mathcal{W}_0 := \{\tilde{g} \in \mathcal{W} : supp(\tilde{g}) \cap \Omega_t \neq \emptyset \text{ only for a finite number of } t \in \Gamma\}.$$

Remark 3.3. The local-to-global argument used in this article is based on the possibility of having a decomposition of the functions in $\mathcal{W} \subset L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$ as the sum of a collection of functions with support in the extended Whitney cubes and satisfying the additional properties detailed in Lemma 3.4. By using the subspace $\mathcal{W}_0$ instead of $\mathcal{W}$ the sums in the decompositions are finite simplifying the proof of the main result.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain and $\{\Omega_t\}_{t \in \Gamma}$ a decomposition of extended Whitney cubes. Then, there exists a positive constant C_0 such that for any $\tilde{g} \in \mathcal{W}_0$, there is a collection of functions $\{\tilde{g}_t\}_{t \in \Gamma}$ with the following properties:*

(1) $\tilde{g} = \sum_{t \in \Gamma} \tilde{g}_t$.
(2) $supp(\tilde{g}_t) \subset \Omega_t$.
(3) $\tilde{g}_t \in \mathcal{W}_0$, *for all $t \in \Gamma$.*

We call this collection of functions a $\mathcal{V}$-decomposition of $\tilde{g}$ subordinate to $\{\Omega_t\}_{t \in \Gamma}$.
In addition, it satisfies

$$(3.7) \qquad \sum_{t \in \Gamma} \|\tilde{g}_t\|^q_{L^q(\Omega_t, \rho^{-q\beta})} \leq C_0^q \|\tilde{g}\|^q_{L^q(\Omega, \rho^{-q\beta})}.$$

Moreover, $\tilde{g}_t \not\equiv 0$ only for a finite number of $t \in \Gamma$.

Lemma 3.4, which is fundamental in these notes, is proved in the next section.

Now, we define the following subspace $\mathcal{S}$ of $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$ by

$$(3.8) \qquad\qquad\qquad \mathcal{S} := \mathcal{W}_0 \oplus \rho^{p\beta} \mathcal{V}.$$

Lemma 3.5. *The subspace $\mathcal{S}$ defined above is dense in $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$.*

Proof. By Lemma 3.2, it is sufficient to show that $\mathcal{W}_0$ is dense in $\mathcal{W}$ with respect to the norm in $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$.

Let $Q \subset \Omega$ be a cube that intersects a finite collection of Ω_t and $\{\tilde{\nu}_j\}_{1 \le j \le m}$ an orthogonal basis of the finite dimensional space $\mathcal{V}$ with respect to the inner product

$$\langle \tilde{\psi}, \tilde{\varphi} \rangle_Q = \int_Q \tilde{\psi}(x) : \tilde{\varphi}(x)\, \rho^{p\beta}(x)\, \mathrm{d}x.$$

Now, given $\tilde{h} \in \mathcal{W}$ notice that $\int_\Omega \tilde{h} : \tilde{\nu}_j = 0$ for all j, for being $\tilde{\nu}_j$ a function in $\mathcal{V}$. Next, given $\epsilon > 0$, let $\Omega_\epsilon \subset \Omega$ be an open set that contains Q, intersects a finite number of Ω_t and

$$\|(1 - \chi_{\Omega_\epsilon})\tilde{h}\|_{L^q(\Omega, \rho^{-q\beta})} < \epsilon.$$

Thus, we define the function $\tilde{g}$ by

$$\tilde{g}(x) := \chi_{\Omega_\epsilon}(x)\tilde{h}(x) + \sum_{i=1}^{m} \chi_Q(x)\rho^{p\beta}(x)\tilde{\nu}_i(x) \int_{\Omega \setminus \Omega_\epsilon} \tilde{h}(y) : \tilde{\nu}_i(y)\, \mathrm{d}y.$$

Notice first that the support of $\tilde{g}$ intersects a finite number of Ω_t, and $\int_\Omega \tilde{g} : \tilde{\nu}_j = 0$ for all j, thus $\tilde{g}$ belongs to $\mathcal{W}_0$. Moreover,

$$
\begin{aligned}
\|\tilde{h} - \tilde{g}\|_{L^q(\Omega, \rho^{-q\beta})} &\le \epsilon + \sum_{i=1}^{m} \left\| \chi_Q(x)\rho^{p\beta}(x)\tilde{\nu}_i(x) \int_{\Omega \setminus \Omega_\epsilon} \tilde{h}(y) : \tilde{\nu}_i(y)\, \mathrm{d}y \right\|_{L^q(\Omega, \rho^{-q\beta})} \\
&\le \epsilon + \sum_{i=1}^{m} \left(\int_{\Omega \setminus \Omega_\epsilon} |\tilde{h}(y) : \tilde{\nu}_i(y)|\, \mathrm{d}y \right) \|\tilde{\nu}_i \rho^{p\beta}\|_{L^q(Q, \rho^{-q\beta})} \\
&\le \epsilon + \sum_{i=1}^{m} \|\tilde{h}\|_{L^q(\Omega \setminus \Omega_\epsilon, \rho^{-q\beta})} \|\tilde{\nu}_i\|_{L^p(\Omega, \rho^{p\beta})} \|\tilde{\nu}_i \rho^{p\beta}\|_{L^q(Q, \rho^{-q\beta})} \\
&\le \epsilon \left(1 + \sum_{i=1}^{m} \|\tilde{\nu}_i\|_{L^p(\Omega, \rho^{p\beta})} \|\tilde{\nu}_i \rho^{p\beta}\|_{L^q(Q, \rho^{-q\beta})} \right),
\end{aligned}
$$

which proves that $\mathcal{S}$ is dense in $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$. $\qquad\qquad \square$

We are ready to prove the main result of this article.

Proof of Theorem 1.1. Let $\mathbf{v}$ be an arbitrary vector field in $W^{1,p}(\Omega, \mathbb{R}^n, \rho^{p\beta})$. Next, let us take $\mathbf{w}$ in Σ, the kernel of l, such that

$$\int_\Omega (D\mathbf{v} : \tilde{\varphi})\, \rho^{p\beta} = \int_\Omega (D\mathbf{w} : \tilde{\varphi})\, \rho^{p\beta}$$

for all $\tilde{\varphi} \in \mathcal{V}$. Recall that the elements in $\mathcal{V}$ are the differential matrix of the vector fields in (3.2), thus $D\mathbf{w}$ is the orthogonal projection of $D\mathbf{u}$ onto $\mathcal{V}$ with respect to

the inner product used above. Moreover, $\beta \geq 0$ and Ω is bounded, then $\mathbf{w}$ belongs to $W^{1,p}(\Omega, \mathbb{R}^n, \rho^{p\beta})$. Hence, by taking $\mathbf{u} := \mathbf{v} - \mathbf{w}$, it is sufficient to prove

$$\left(\int_\Omega |D\mathbf{u}|^p \rho^{p\beta} \, \mathrm{d}x \right)^{1/p} \leq C \left(\int_\Omega |l(\mathbf{u})|^p \rho^{p\beta} \, \mathrm{d}x \right)^{1/p},$$

for all $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n, \rho^{p\beta})$ which satisfy

$$\int_\Omega (D\mathbf{u} : \tilde{\varphi}) \, \rho^{p\beta} = 0$$

for all $\tilde{\varphi} \in \mathcal{V}$. For simplicity, we preferred to write the generalized version of the Korn inequality in our main theorem by using the infimum over $l(\mathbf{w}) = 0$, however, it is also valid for vector fields verifying the condition stated above, which is also very standard in this kind of inequalities.

Now, using that the space $\mathcal{S}$ defined in (3.8) is dense in $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta})$, it is enough to show that there is a constant C such that

$$\int D\mathbf{u} : (\tilde{g} + \rho^{p\beta}\tilde{\psi}) \leq C \left(\int_\Omega |l(\mathbf{u})|^p \rho^{p\beta} \, \mathrm{d}x \right)^{1/p},$$

for any arbitrary function $\tilde{g} + \rho^{p\beta}\tilde{\psi}$ in $\mathcal{S}$, with $\|\tilde{g} + \rho^{p\beta}\tilde{\psi}\|_{L^q(\Omega, \rho^{-q\beta})} \leq 1$. Thus, given a function $\tilde{g} + \rho^{p\beta}\tilde{\psi}$ with norm less than one, let us take a $\mathcal{V}$-decomposition $\{\tilde{g}_t\}_{t \in \Gamma}$ of $\tilde{g}$ (we refer to Lemma 3.4). Thus,

$$\int_\Omega D\mathbf{u} : (\tilde{g} + \rho^{p\beta}\tilde{\psi}) = \int_\Omega D\mathbf{u} : \tilde{g}$$
$$= \int_{\Omega_t} D\mathbf{u} : \sum_{t \in \Gamma} \tilde{g}_t$$
$$= \sum_{t \in \Gamma} \int_{\Omega_t} D\mathbf{u} : \tilde{g}_t = (1).$$

Notice that in the last identity we used the finiteness of the sum stated in Lemma 3.4. Now, $\tilde{g}_t$ satisfies that $\int_{\Omega_t} D\mathbf{w} : \tilde{g}_t = 0$ for all $\mathbf{w} \in \Sigma$. Thus, from Hölder inequality, property (2) in Whitney decomposition's definition and (3.1) we obtain

$$(1) \leq \sum_{t \in \Gamma} \inf_{l(\mathbf{w})=0} \|D\mathbf{u} - D\mathbf{w}\|_{L^p(\Omega_t, \rho^{p\beta})} \|\tilde{g}_t\|_{L^q(\Omega_t, \rho^{-q\beta})}$$
$$\leq \sum_{t \in \Gamma} C(\mathrm{diam}(\Omega_t))^\beta \inf_{l(\mathbf{w})=0} \|D\mathbf{u} - D\mathbf{w}\|_{L^p(\Omega_t)} \|\tilde{g}_t\|_{L^q(\Omega_t, \rho^{-q\beta})}$$
$$\leq C \sum_{t \in \Gamma} (\mathrm{diam}(Q_t))^\beta \|l(\mathbf{u})\|_{L^p(\Omega_t)} \|\tilde{g}_t\|_{L^q(\Omega_t, \rho^{-q\beta})}$$
$$\leq C \sum_{t \in \Gamma} \|l(\mathbf{u})\|_{L^p(\Omega_t, \rho^{p\beta})} \|\tilde{g}_t\|_{L^q(\Omega_t, \rho^{-q\beta})} = (2).$$

Next, we use Hölder inequality for the sum depending on t to obtain

$$(2) \leq C \left(\sum_{t \in \Gamma} \int_{\Omega_t} |l(\mathbf{u})|^p \rho^{p\beta} \right)^{1/p} \left(\sum_{t \in \Gamma} \|\tilde{g}_t\|_{L^q(\Omega_t, \rho^{-q\beta})}^q \right)^{1/q} = (3).$$

Using that each cube Ω_t intersects no more than 12^n cubes in $\{\Omega_s\}_{s\in\Gamma}$, and Lemma 3.4 we conclude

$$(3) \leq C\,C_0\,\|l(\mathbf{u})\|_{L^p(\Omega,\rho^{p\beta})}\|\tilde{g}\|_{L^q(\Omega,\rho^{-q\beta})} \leq C\,C_0\,C_1\|l(\mathbf{u})\|_{L^p(\Omega,\rho^{p\beta})},$$

where C is independent of Ω, C_0 is the estimate in (3.7) and C_1 is the constant in (3.4). $\qquad\square$

In the following two corollaries we show that the Korn type inequalities considered in [15, 16] are valid on bounded John domains.

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain, with $n \geq 3$, $1 < p < \infty$ and $\Pi : W^{1,p}(\Omega,\mathbb{R}^n) \to \Sigma$ a projection, where Σ is endowed with the topology of $L^p(\Omega,\mathbb{R}^n)$. Then, there is a constant C such that*

$$\|\mathbf{v} - \Pi(\mathbf{v})\|_{W^{1,p}(\Omega)} \leq C\|l(\mathbf{v})\|_{L^p(\Omega)},$$

for all $\mathbf{v} \in W^{1,p}(\Omega,\mathbb{R}^n)$.

Proof. Let $\mathbf{w}_0 \in \Sigma$ be such that

$$\|D\mathbf{v} - D\mathbf{w}_0\|_{L^p(\Omega)} \leq 2\inf_{l(\mathbf{w})=0}\|D\mathbf{v} - D\mathbf{w}\|_{L^p(\Omega)}$$

and $\int_\Omega \mathbf{v}_i - \mathbf{w}_{0i} = 0$ for all $1 \leq i \leq n$. Then, by using that Π is a projection and the norms $\|\cdot\|_{W^{1,p}(\Omega)}$ or $\|\cdot\|_{L^p(\Omega)}$ are equivalent over Σ for being Σ a finite dimensional space, we have

$$\begin{aligned}
\|\mathbf{v} - \Pi(\mathbf{v})\|_{W^{1,p}(\Omega)} &\leq \|\mathbf{v} - \mathbf{w}_0\|_{W^{1,p}(\Omega)} + \|\Pi(\mathbf{v}) - \Pi(\mathbf{w}_0)\|_{W^{1,p}(\Omega)} \\
&\leq C\{\|\mathbf{v} - \mathbf{w}_0\|_{W^{1,p}(\Omega)} + \|\Pi(\mathbf{v}) - \Pi(\mathbf{w}_0)\|_{L^p(\Omega)}\} \\
&\leq C\|\mathbf{v} - \mathbf{w}_0\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

Finally, by using Poincaré inequality on Ω (see for example [20]) and Theorem 1.1 with $\beta = 0$ we conclude

$$\begin{aligned}
\|\mathbf{v} - \Pi(\mathbf{v})\|_{W^{1,p}(\Omega)} &\leq C\|D\mathbf{v} - D\mathbf{w}_0\|_{L^p(\Omega)} \\
&\leq C\inf_{l(\mathbf{w})=0}\|D\mathbf{v} - D\mathbf{w}\|_{L^p(\Omega)} \\
&\leq C\|l(\mathbf{v})\|_{L^p(\Omega)}.
\end{aligned}$$

$\square$

Corollary 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain, with $n \geq 3$. Then, there is a constant C such that*

$$(3.9) \qquad\qquad \|\mathbf{v}\|_{W^{1,2}(\Omega)} \leq C\{\|\mathbf{v}\|_{L^2(\Omega)} + \|l(\mathbf{v})\|_{L^2(\Omega)}\},$$

for all $\mathbf{v} \in W^{1,2}(\Omega,\mathbb{R}^n)$.

Proof. Let Q be a cube included in Ω and $\Pi_Q : W^{1,2}(Q,\mathbb{R}^n) \to \Sigma$ a projection. The norms $\|\cdot\|_{L^2(Q)}$ and $\|\cdot\|_{L^2(\Omega)}$ are equivalent over Σ, thus Π_Q is also a projection from $W^{1,2}(\Omega,\mathbb{R}^n)$ to Σ. Hence, using Corollary 3.6 over Ω and the validity of (3.9) over the

cube Q, we show for any $\mathbf{v}$ in $W^{1,2}(\Omega, \mathbb{R}^n)$ that

$$
\begin{aligned}
\|\mathbf{v}\|_{W^{1,2}(\Omega)} &\leq \|\mathbf{v} - \Pi_Q(\mathbf{v})\|_{W^{1,2}(\Omega)} + \|\Pi_Q(\mathbf{v})\|_{W^{1,2}(\Omega)} \\
&\leq C\{\|l(\mathbf{v})\|_{L^2(\Omega)} + \|\Pi_Q(\mathbf{v})\|_{L^2(Q)}\} \\
&\leq C\{\|l(\mathbf{v})\|_{L^2(\Omega)} + \|\mathbf{v}\|_{W^{1,2}(Q)}\} \\
&\leq C\{\|l(\mathbf{v})\|_{L^2(\Omega)} + \|\mathbf{v}\|_{L^2(Q)} + \|l(\mathbf{v})\|_{L^2(Q)}\} \\
&\leq C\{\|\mathbf{v}\|_{L^2(Q)} + \|l(\mathbf{v})\|_{L^2(\Omega)}\}
\end{aligned}
$$

concluding the proof. $\square$

To finalize this section we show an example that proves that the generalized Korn inequality (1.5) might fail if Ω is not a John domain. Thus, let us consider the case $p = 2$, $\beta = 0$ and the cuspidal domain $\Omega \subset \mathbb{R}^3$ given by

$$
\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, 0 < x_1, x_2 < 1 \text{ and } 0 < x_3 < x_2^\gamma\},
$$

with $\gamma > 1$. Let us assume by contradiction that (1.5) holds on Ω, thus, following the ideas in the previous two corollaries, we can conclude that there is a constant C such that

$$
(3.10) \qquad \|D\mathbf{v}\|_{L^2(\Omega)} \leq C\{\|\mathbf{v}\|_{L^2(Q)} + \|l(\mathbf{v})\|_{L^2(\Omega)}\},
$$

for all $\mathbf{v} \in W^{1,2}(\Omega, \mathbb{R}^3)$, where Q is a fixed cube in Ω. Now, let us consider the vector field

$$
\mathbf{v}(x_1, x_2, x_3) := (0, -(s+1)x_3 x_2^s, x_2^{s+1}).
$$

By a straightforward calculation it can be seen that if s satisfies that

$$
\max\left\{-\frac{(\gamma+1)}{2} - (\gamma - 1), -\frac{(\gamma+1)}{2} - 1\right\} < s < -\frac{(\gamma+1)}{2}
$$

then the left hand side of (3.10) is infinite while the right one is finite following in a contradiction.

This kind of counterexamples has been studied in [1] to show that the Korn inequality

$$
(3.11) \qquad \|D\mathbf{v}\|_{L^2(\Omega)} \leq C\{\|\mathbf{v}\|_{L^2(\Omega)} + \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)}\},
$$

fails on certain cuspidal domains of the style of Ω. The fact that this counterexample also works for the generalized version of Korn can also be concluded by observing that

$$
\|l(\mathbf{v})\|_{L^2(\Omega)}^2 = \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)}^2 - \frac{1}{n}\|\operatorname{div}\mathbf{v}\|_{L^2(\Omega)}^2 \leq \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)}^2
$$

which implies that (3.10) fails when (3.11) does.

Remark 3.8. Notice that the counterexample described above is not sufficient to conclude that the validity of the weighted generalized Korn inequality (1.5) implies that the domain is John. The equivalence between the Korn inequality (in the second case) and John's condition is studied in [8, 21], however, the equivalence with the generalized Korn inequality considered in the main result has not been proved yet.

4. $\mathcal{V}$-DECOMPOSITION OF FUNCTIONS

The $\mathcal{V}$-decomposition of functions introduced in Lemma 3.4 is constructed by using an argument based on a certain partial order on the Whitney cubes $\{Q_t\}_{t\in\Gamma}$.

Let us denote by $G = (V, E)$ a graph with vertices V and edges E. Graphs in these notes do not have neither multiple edges nor loops and the number of vertices in V is countable. Moreover, each vertex is of finite degree, i.e. only a finite number of vertices emanate from each vertex. A *rooted tree* (or simply a tree) is a connected graph $G = (V, E)$ in which any two vertices are connected by exactly one simple path, and a *root* is simply a distinguished vertex $a \in V$. Moreover, if $G = (V, E)$ is a rooted tree, it is possible to define a *partial order* "$\preceq$" in V as follows: $s \preceq t$ if and only if the unique path connecting t with the root a passes through s. The *height* or *level* of any $t \in V$ is the number of vertices in $\{s \in V : s \preceq t \text{ with } s \neq t\}$. *The parent* of a vertex $t \in V$ is the vertex s satisfying that $s \preceq t$ and its height is one unit smaller than the height of t. We denote the parent of t by t_p. It can be seen that each $t \in V$ different from the root has a unique parent, but several elements in V could have the same parent. Note that two vertices are connected by an edge (*adjacent vertices*) if one is the parent of the other.

Now, if $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\{Q_t\}_{t\in\Gamma}$ a Whitney decomposition, we define the connected graph

$$G_\Gamma = (\Gamma, E_\Gamma)$$

where the set of vertices is the set of subindexes Γ and two arbitrary s, t in Γ are connected by an edge iff Q_s and Q_t are $(n-1)$- neighbors.

Definition 4.1. A *tree structure* of Γ is given by a collection of edges $E \subset E_\Gamma$ and a distinguished vertex $a \in \Gamma$ such that the subgraph $G = (\Gamma, E)$ of G_Γ is a rooted tree.

There are different tree structures that can be added to Γ. For example, we can define one such that the path that connects each vertex t with the root a has minimal length. This kind of paths are not unique thus the tree structure must be defined inductively, level by level, by choosing a path with minimal length. This example was considered in [22] where the tree structure added to Γ is related to the quasi-hyperbolic distance in Ω. Specifically, the quasi-hyperbolic distance between Q_t and Q_a in Ω is comparable to the length of the path that connects t with a in Γ, see [23] for details.

In the following picture, we sketch another example that shows some cubes of a Whitney decomposition of a circle and how a tree structure looks like.

Definition 4.2. Given a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ of a domain $\Omega \subset \mathbb{R}^n$ and a tree structure of Γ, we take a collection of open pairwise disjoint cubes $\{B_t\}_{t\neq a}$ with sides parallel to the axis such that $B_t \subseteq \Omega_t \cap \Omega_{t_p}$ and $|\Omega_t| \leq C_n |B_t|$ for all $t \in \Gamma$, where $\{\Omega_t\}_{t\in\Gamma}$ are extended Whitney cubes associated to $\{Q_t\}_{t\in\Gamma}$. This collection of cubes exists by following the properties for Whitney cubes described in Section 2. Thus, using the tree structure of Γ, we define *the Hardy type operator T* for functions in $L^1(\Omega)$ by:

$$(4.1) \qquad Tg(x) := \sum_{a \neq t \in \Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |g|,$$

where $\chi_t(x)$ is the characteristic function of B_t, for all $t \neq a$, and $W_t = \bigcup_{s \succeq t} \Omega_s$.
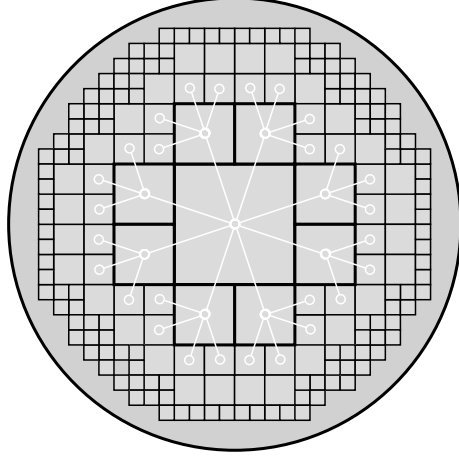
FIGURE 1. Some Whitney cubes Q_t of a circle and a tree structure

We refer to W_t by *the shadow of Ω_t*. This is a fairly known name and it follows the assumption that light travels from Ω_a to the different cubes Ω_t along the unique path that connects them. This geometric interpretation was taken from [24], page 81, in the context of quasi-hyperbolic geodesics and chains of Whitney cubes with minimal number of cubes.

The following lemma, shown in the companion article [5], describes the geometry of John domains in terms of Whitney decompositions and trees. The articles [25] and [26] were fundamental in the proof of this lemma.

Lemma 4.3. *Given a bounded John domain Ω and a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$, there exists a constant $K > 1$ and a tree structure of Γ, with root "a", that satisfies*

(4.2) $$Q_s \subseteq KQ_t,$$

for any $s, t \in \Gamma$ with $s \succeq t$. In other words, the shadow of Q_t is contained in KQ_t.

From now on $\Omega \subset \mathbb{R}^n$ is a bounded John domain, $\{\Omega_t\}_{t\in\Gamma}$ is the collection of extended Whitney cubes defined in Section 2 and Γ has a tree structure with the geometric property introduced in Lemma 4.3. The following lemma is proved in [5].

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain, $\beta \geq 0$ and $1 < q < \infty$. Then, the operator T defined in (4.1) is continuous from $L^q(\Omega, \rho^{-q\beta})$ to itself, where ρ is the distance to the boundary of Ω. Moreover, its norm is bounded by*

$$\|T\|_{L\to L} \leq CK^\beta,$$

where L denotes $L^q(\Omega, \rho^{-q\beta})$. The constant C in the previous inequality is independent of Ω and K is the constant introduced in (4.2).

Now, we are ready to construct the $\mathcal{V}$-decomposition.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain and $\{\Omega_t\}_{t\in\Gamma}$ a collection of extended Whitney cubes. Given $\tilde{g} \in L^1(\Omega, \mathbb{R}^{n\times n})$ such that $\int_\Omega \tilde{g} : \tilde{\varphi} = 0$, for all $\tilde{\varphi} \in \mathcal{V}$, and $\mathrm{supp}(\tilde{g}) \cap \Omega_s \neq \emptyset$ only for a finite number of $s \in \Gamma$, there exists a collection of functions $\{\tilde{g}_t\}_{t\in\Gamma}$ in $L^1(\Omega, \mathbb{R}^{n\times n})$ with the following properties:*

(1) $\tilde{g} = \sum_{t \in \Gamma} \tilde{g}_t$.
(2) $supp(\tilde{g}_t) \subset \Omega_t$.
(3) $\int_{\Omega_t} \tilde{g}_t : \tilde{\varphi} = 0$, for all $\tilde{\varphi} \in \mathcal{V}$.

Moreover, let $t \in \Gamma$. If $x \in B_s$, where $s = t$ or $s_p = t$, we have the following pointwise estimate

$$(4.3) \qquad \|\tilde{g}_t\|_\infty(x) \le \|\tilde{g}\|_\infty(x) + C_n K^{n+1} T \|\tilde{g}\|_1(x),$$

where K is the geometric constant introduced in (4.2) and C_n is a constant that depends only on n. Otherwise, if $x \notin \bigcup_{s \in \Gamma} B_s$ or $x \in B_s$, where $s \ne t$ and $s_p \ne t$, then

$$(4.4) \qquad \|\tilde{g}_t\|_\infty(x) \le \|\tilde{g}\|_\infty(x).$$

Finally, $\tilde{g}_t \equiv 0$ for all $t \in \Gamma \setminus \Gamma_{\tilde{g}}$, where $\Gamma_{\tilde{g}}$ is the subtree of Γ with a finite number of vertices given by

$$\Gamma_{\tilde{g}} := \{s \in \Gamma : \text{ there is } k \succeq s \text{ with } supp(\tilde{g}) \cap \Omega_k \ne \emptyset\}.$$

Proof. Let us define a base of the vector space $\mathcal{V}$. For the constant skew-symmetric matrices we consider

$$(E_{ij})_{i'j'} = \begin{cases} 1 & \text{if } (i', j') = (i, j) \\ -1 & \text{if } (i', j') = (j, i) \\ 0 & \text{otherwise} \end{cases}$$

where $1 \le i < j \le n$. It can be seen that the dimension of $\mathcal{V}$ equals $n_0 + n$, where $n_0 := \frac{n(n-1)}{2} + 1$. Let us take the following basis $\{A_1, \cdots, A_{n_0}, H_1(x), \cdots, H_n(x)\}$ of $\mathcal{V}$, where the first $\frac{n(n-1)}{2}$ elements are the matrices with constant coefficients E_{ij}, following an arbitrary order, and $A_{n_0} = I$ is the identity matrix. The matrices $H_i(x)$ have been previously defined in (3.3).

Now, let $\{\phi_t\}_{t \in \Gamma}$ be a partition of the unity subordinate to $\{\Omega_t\}_{t \in \Gamma}$. Namely, a collection of smooth functions such that $\sum_{t \in \Gamma} \phi_t = 1$, $0 \le \phi_t \le 1$ and $supp(\phi_t) \subset \Omega_t$. Thus, $\tilde{g}$ can be cut-off into $\tilde{g} = \sum_{t \in \Gamma} \tilde{f}_t$ by taking $\tilde{f}_t(x) = \phi(x)\tilde{g}(x)$. Note that $\tilde{f}_t \equiv 0$ except for a finite number of $t \in \Gamma$. This decomposition satisfies (1) and (2) in the statement of the lemma but not necessarily (3). Thus, we make some modifications to obtain the orthogonality with respect to $\mathcal{V}$. We construct the decomposition in two steps. We deal first with the orthogonality with respect to the matrices with constant coefficients $\{A_1, \cdots, A_{n_0}\}$ and later with respect to $\{H_1(x), \cdots, H_n(x)\}$.

First step: The decomposition in this first step is denoted with the upper index (0). Thus, we define the functions $A_{i,s}(x)$ as a sort of normalization of A_i with respect to a certain inner product over B_s:

$$A_{i,s}(x) := \frac{\chi_s(x) A_i}{2|B_s|} \qquad \text{if } 1 \le i \le n_0 - 1$$

$$A_{n_0,s}(x) := \frac{\chi_s(x) A_{n_0}}{n|B_s|} = \frac{\chi_s(x) I}{n|B_s|},$$

where $\chi_s(x)$ is the characteristic function of B_s. Indeed, notice that $\int A_{i,s}(x) : A_j \, \mathrm{d}x = \delta_{i,j}$ for all $s \in \Gamma \setminus \{a\}$ and $1 \le i, j \le n_0$, where $\delta_{i,j}$ is the Kronecker delta.

Thus, we define the collection of functions $\{\tilde{g}_t^{(0)}\}_{t\in\Gamma}$ from Ω to $\mathbb{R}^{n\times n}$ by

$$(4.5) \qquad \tilde{g}_t^{(0)}(x) := \tilde{f}_t(x) + \left(\sum_{s:\, s_p=t} \tilde{h}_s^{(0)}(x)\right) - \tilde{h}_t^{(0)}(x),$$

where

$$(4.6) \qquad \tilde{h}_s^{(0)}(x) := \sum_{i=1}^{n_0}\left(\int_{W_s} A_i : \sum_{k\succeq s}\tilde{f}_k(y)\,\mathrm{d}y\right)A_{i,s}(x).$$

The sum in (4.5) is indexed over every $s \in \Gamma$ such that t is the parent of s. In the particular case when t is the root of Γ, (4.5) means

$$\tilde{g}_a^{(0)}(x) = \tilde{f}_a(x) + \sum_{s:\, s_p=a}\tilde{h}_s^{(0)}(x).$$

Notice that the functions $\tilde{h}_s^{(0)}$ in (4.6) are well-defined because of the integrability of $\tilde{g}$. Indeed,

$$\left|\int_{W_s} A_i : \sum_{k\succeq s}\tilde{f}_k(y)\,\mathrm{d}y\right| \le \|A_i\|_\infty \int_{W_s}\left\|\sum_{k\succeq s}\tilde{f}_k\right\|_1(y)\,\mathrm{d}y$$

$$\le \int_{W_s}\|\tilde{g}\|_1(y)\,\mathrm{d}y.$$

See definitions of $\|\tilde{g}\|_r(y)$, for $1 \le r \le \infty$, in Section 2. Moreover, it can be easily observed that $\tilde{f}_s$, $\tilde{h}_s^{(0)}$ and $\tilde{g}_s^{(0)}$ are identically zero for all $s \in \Gamma \setminus \Gamma_{\tilde{g}}$. For this reason the sums indexed over subsets of Γ, for instance $k \succeq s$, that appear in the first step are finite. The finiteness of these sums is also verified in the second step.

We know that

$$supp(\tilde{h}_s^{(0)}) \subset B_s$$

and the coefficients of $\tilde{h}_s^{(0)}(x)$ are estimated in the following way:

$$(4.7) \qquad \|\tilde{h}_s^{(0)}\|_\infty(x) \le \tfrac{1}{2|B_s|}\chi_s(x)\int_{W_s}\|\tilde{g}\|_1(y)\,\mathrm{d}y = \tfrac{|W_s|}{2|B_s|}\chi_s(x)T\|\tilde{g}\|_1(x)$$

for all $x \in \Omega$. Thus, using (4.2), if $x \in B_s$ where $s = t$ or $s_p = t$ then

$$(4.8) \qquad \begin{aligned} \|\tilde{g}_t^{(0)}\|_\infty(x) &\le \|\tilde{g}\|_\infty(x) + \tfrac{|W_s|}{2|B_s|}T\|\tilde{g}\|_1(x)\\ &\le \|\tilde{g}\|_\infty(x) + K^n\,T\|\tilde{g}\|_1(x).\end{aligned}$$

Otherwise, if $x \notin \bigcup_{s\in\Gamma} B_s$ or $x \in B_s$ with $s \neq t$ and $s_p \neq t$ then

$$\|\tilde{g}_t^{(0)}\|_\infty(x) \le \|\tilde{g}\|_\infty(x).$$

Let us continue by showing that $\tilde{g}(x) = \sum_{t\in\Gamma}\tilde{g}_t^{(0)}(x)$ for all x. Given $x \in \Omega$, let suppose that $x \notin \bigcup B_t$. Then $\tilde{g}_t^{(0)}(x) = \tilde{g}_t(x)$ for all $t \in \Gamma$, then

$$\sum_{t\in\Gamma}\tilde{g}_t^{(0)}(x) = \sum_{t\in\Gamma}\tilde{g}_t(x) = \tilde{g}(x).$$

Otherwise, if x belongs to B_{k_0} for $k_0 \in \Gamma$ it follows that $\tilde{g}_t^{(0)}(x) = \tilde{g}_t(x)$ for all $t \neq k_0$, $t \neq k_{0_p}$ (k_{0_p} is the parent of k_0). Moreover, by using that the cubes B_s are pairwise disjoint we have

$$\tilde{g}_{k_0}^{(0)}(x) = \tilde{g}_{k_0}(x) - \tilde{h}_{k_0}^{(0)}(x)$$
$$\tilde{g}_{k_{0_p}}^{(0)}(x) = \tilde{g}_{k_{0_p}}(x) + \tilde{h}_{k_0}^{(0)}(x).$$

Then, $\sum_{t \in \Gamma} \tilde{g}_t^{(0)}(x) = \tilde{g}(x)$ for all x.

Next, let us prove (2) in the statement of the lemma. The parent of each s in the sum in (4.5) is t, then $B_s \subseteq \Omega_s \cap \Omega_t$. Thus, $supp(\tilde{g}_t^{(0)}) \subseteq \Omega_t$.

Now, let us show property (3), which refers to the orthogonality of $\tilde{g}_t^{(0)}$ with respect to the matrices $A_1, \cdots, A_{n_0}$ for all $t \in \Gamma$. Observe that $k \succeq t$ if and only if $k \succeq s$, with $s_p = t$, or $k = t$. Thus, given $1 \leq j \leq n_0$

$$\int \tilde{h}_t^{(0)}(x) : A_j \, dx = \int A_j : \sum_{k \succeq t} \tilde{f}_k(y) \, dy$$
$$= \int A_j : \tilde{f}_t(y) \, dy + \sum_{s : s_p = t} \int A_j : \sum_{k \succeq s} \tilde{f}_k(y) \, dy$$
$$= \int A_j : \tilde{f}_t(y) \, dy + \sum_{s : s_p = t} \int \tilde{h}_s^{(0)}(x) : A_j \, dx.$$

Then, $\int \tilde{g}_t^{(0)}(x) : A_j \, dx = 0$, for all $t \neq a$.

Finally,

$$\int \tilde{g}_a^{(0)}(x) : A_j \, dx = \int \tilde{f}_a(x) : A_j \, dx + \sum_{s : s_p = a} \int A_j : \sum_{k \succeq s} \tilde{f}_k(y) \, dy$$
$$= \int A_j : \sum_{k \succeq a} \tilde{f}_k(y) \, dy$$
$$= \int A_j : \tilde{g}(x) \, dx = 0.$$

Second step: In this step the decomposition is denoted with the upper index (1). Now, we repeat the process used in the first step replacing the collection $\{\tilde{f}_t\}_{t \in \Gamma}$ by $\{\tilde{g}_t^{(0)}\}_{t \in \Gamma}$ and the matrices $A_1, \cdots, A_{n_0}$ by $H_1(x), \cdots, H_n(x)$.

Given a cube B_s in Definition 4.2, with $s \in \Gamma \setminus \{a\}$, and $1 \leq i \leq n$ we define

$$\tilde{\theta}_{i,s}(x) := \frac{H_i(x - c_s)\chi_s(x)}{\int_{B_s} H_i(z - c_s) : H_i(z - c_s) \, dz},$$

where c_s is the center of the cube B_s. Using the symmetries of the cubes B_s which have sides parallel to the axis, it follows that $\int \tilde{\theta}_{s,i}(x) : H_j(x) \, dx = \delta_{i,j}$ for all $s \in \Gamma \setminus \{a\}$ and $1 \leq i, j \leq n$. Moreover, notice that $\int \tilde{\theta}_{s,i}(x) : A_j(x) \, dx = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq n_0$. This property is basic to preserve the orthogonality obtained in the first step.

We define the decomposition of g in the following way:

$$(4.9) \qquad \tilde{g}_t(x) := \tilde{g}_t^{(0)}(x) + \left(\sum_{s:\, s_p = t} \tilde{h}_s^{(1)}(x) \right) - \tilde{h}_t^{(1)}(x),$$

where

$$(4.10) \qquad \tilde{h}_s^{(1)}(x) := \sum_{i=1}^{n} \left(\int_{W_s} H_i(y) : \sum_{k \succeq s} \tilde{g}_k^{(0)}(y)\, dy \right) \tilde{\theta}_{s,i}(x).$$

In the particular case when $t = a$, (4.9) means

$$\tilde{g}_a(x) \;\; := \;\; \tilde{g}_a^{(0)}(x) + \sum_{s:\, s_p = a} \tilde{h}_s^{(1)}(x).$$

As before, $\tilde{g}_s^{(0)}$ and $\tilde{h}_s^{(1)}$ are identically zero if $s \in \Gamma \setminus \Gamma_{\tilde{g}}$ implying that $\tilde{g}_s$ is identically zero if $s \in \Gamma \setminus \Gamma_{\tilde{g}}$.

In order to prove (4.4), notice that $supp(\tilde{h}_s^{(0)}) \subseteq B_s$ and $supp(\tilde{h}_s^{(1)}) \subseteq B_s$. Thus, from (4.9) and (4.5), it follows that $\tilde{g}_t(x) = \tilde{f}_t(x)$ for all $x \notin \bigcup_{s \in \Gamma} B_s$ or $x \in B_s$ with $s \neq t$ and $s_p \neq t$. Then, (4.4) is proved.

Estimate (4.3) requires more work. Let us start by showing a pointwise estimate of $\|\tilde{h}_s^{(1)}\|_1(x)$ by the Hardy type operator T on $\|\tilde{g}\|_1(x)$. First, notice that

$$\|\tilde{\theta}_{s,i}\|_\infty(x) \le \frac{6}{l_s^{n+1}} = \frac{6}{l_s |B_s|},$$

where l_s is the side length of the cube B_s. Next, using the orthogonality of the collection $\{\tilde{g}_k^{(0)}\}$ with respect to $A_1, \cdots, A_{n_0}$, we can conclude that

$$\int_{W_s} H_i(y) : \sum_{k \succeq s} \tilde{g}_k^{(0)}(y)\, dy = \int_{W_s} H_i(y - c_s) : \sum_{k \succeq s} \tilde{g}_k^{(0)}(y)\, dy.$$

Thus, by replacing this new integral in definition (4.10) and using that $\|H_i\|_\infty(y - c_s) \le diam(W_s)$ for all $y \in W_s$, we have

$$\begin{aligned}
\|\tilde{h}_s^{(1)}\|_\infty(x) &\le \sum_{i=1}^{n} \|\tilde{\theta}_{s,i}\|_\infty(x) \int_{W_s} |H_i(y - c_s)| : |\sum_{k \succeq s} \tilde{g}_k^{(0)}(y)|\, dy \\
&\le \sum_{i=1}^{n} 6 \frac{\chi_s(x)}{l_s |B_s|} \int_{W_s} \|H_i\|_\infty(y - c_s) \| \sum_{k \succeq s} \tilde{g}_k^{(0)} \|_1(y)\, dy \\
&\le \sum_{i=1}^{n} 6 \frac{diam(W_s)}{l_s} \frac{\chi_s(x)}{|B_s|} \int_{W_s} \| \sum_{k \succeq s} \tilde{g}_k^{(0)} \|_1(y)\, dy \\
&= 6n \frac{diam(W_s)}{l_s} \frac{\chi_s(x)}{|B_s|} \int_{W_s} \| \sum_{k \succeq s} \tilde{g}_k^{(0)} \|_1(y)\, dy = (1)
\end{aligned}$$

Now, it can be seen by using (4.5) and certain telescopic cancelations of the functions $\tilde{h}_{k'}^{(0)}$ that

$$\sum_{k \succeq s} \tilde{g}_k^{(0)} = \sum_{k \succeq s} \left(\tilde{f}_k + \left(\sum_{k' : k_p' = k} \tilde{h}_{k'}^{(0)} \right) - \tilde{h}_k^{(0)} \right) = \left(\sum_{k \succeq s} \tilde{f}_k \right) - \tilde{h}_s^{(0)}.$$

Thus, from (4.7) it follows

$$(1) = 6n \frac{\text{diam}(W_s)}{l_s} \frac{\chi_s(x)}{|B_s|} \int_{W_s} \| \sum_{k \succeq s} \tilde{f}_k - \tilde{h}_s^{(0)} \|_1 (y) \, dy$$

$$\leq 6n \frac{\text{diam}(W_s)}{l_s} \frac{\chi_s(x)}{|B_s|} \int_{W_s} \| \sum_{k \succeq s} \tilde{f}_k \|_1 (y) + \|\tilde{h}_s^{(0)}\|_1 (y) \, dy$$

$$\leq 6n \frac{\text{diam}(W_s)}{l_s} \frac{\chi_s(x)}{|B_s|} \int_{W_s} \|\tilde{g}\|_1 (y) + \frac{n^2}{2|B_s|} \chi_s(y) \left(\int_{W_s} \|\tilde{g}\|_1(x) \, dx \right) dy$$

$$= 6n \left(1 + \frac{n^2}{2} \right) \frac{\text{diam}(W_s)}{l_s} \frac{|W_s|}{|B_s|} \frac{\chi_s(x)}{|W_s|} \int_{W_s} \|\tilde{g}(y)\|_1 \, dy$$

$$= C_n \frac{\text{diam}(W_s)}{l_s} \frac{|W_s|}{|B_s|} T\|\tilde{g}\|_1(x).$$

Hence, using (4.2),

$$(4.11) \qquad\qquad \|\tilde{h}_s^{(1)}\|_\infty (x) \leq C_n K^{n+1} T\|\tilde{g}\|_1(x).$$

Finally, we have already mentioned that the functions $\tilde{h}_s^{(1)}$ and $\tilde{h}_t^{(1)}$ defined in (4.9) are supported, respectively, in the pairwise disjoint sets B_s and B_t. Thus, combining (4.8) and (4.11), we have that for any $x \in B_s$, where $s = t$ or $s_p = t$,

$$\|\tilde{g}_t\|_\infty (x) \leq \|\tilde{g}^{(0)}\|_\infty (x) + \|\tilde{h}_s^{(1)}\|_\infty (x)$$
$$\leq \|\tilde{g}\|_\infty (x) + K^n T\|\tilde{g}\|_1(x) + C_n K^{n+1} T\|\tilde{g}\|_1(x),$$

proving (4.3).

Properties (1), (2), and (3) in the statement of this lemma follows by using the construction of the partition. Indeed, the first two properties follow by replacing $\tilde{g}_t$ by $\tilde{g}_t^{(0)}$ and $\tilde{g}_t^{(0)}$ by $\tilde{g}_t^{(1)}$ in the first step.

The third property follows from the facts that $\int \tilde{h}_s^{(1)} : A_i = 0$ for all $s \in \Gamma$ and $1 \leq i \leq n_0$, so we do not modify the orthogonality obtained in the previous step, and

$$\int H_j(x) : \tilde{h}_s^{(1)}(x) \, dx = \int H_j(x) : \sum_{k \succeq s} \tilde{g}_k^{(0)}(x) \, dx.$$

The rest of the proof follows by mimicking the first step. □

Proof of Lemma 3.4. As $\beta \geq 0$ and Ω is bounded, $L^q(\Omega, \mathbb{R}^{n \times n}, \rho^{-q\beta}) \subset L^1(\Omega, \mathbb{R}^{n \times n})$. Thus, having just proved Lemma 4.5 we only need to show inequality (3.7). Hence, from (4.3) and (4.4), and the continuity of the operator T stated in Lemma 4.4, we have

$$\sum_{t \in \Gamma} \|\tilde{g}_t\|^q_{L^q(\Omega_t, \rho^{-q\beta})} = \sum_{t \in \Gamma} \int_{\Omega_t} \|\tilde{g}_t\|^q_q(x) \rho^{-q\beta}(x) \, \mathrm{d}x$$

$$= CK^{q(n+1)} \sum_{t \in \Gamma} \int_{\Omega_t} \left(\|\tilde{g}\|^q_q(x) + (T\|\tilde{g}\|_1(x))^q \right) \rho^{-q\beta}(x) \, \mathrm{d}x$$

$$= CK^{q(n+1)} \left(\|\tilde{g}\|^q_{L^q(\Omega, \rho^{-q\beta})} + \|T\|\tilde{g}\|_1\|^q_{L^q(\Omega, \rho^{-q\beta})} \right)$$

$$= CK^{q(n+1+\beta)} \|\tilde{g}\|^q_{L^q(\Omega, \rho^{-q\beta})}$$

where C is independent of Ω. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA RIVERSIDE, 900 UNIVERSITY AVE. RIVERSIDE (92521), CA, USA.

CURRENT ADDRESS: DEPARTMENT OF MATHEMATICS AND STATISTICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY POMONA, 3801 WEST TEMPLE AVENUE, POMONA, CA (91768), USA.

E-mail address: fal@cpp.edu