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Weighted Korn inequalities on John domains

by

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Abstract. We show a weighted version of the Korn inequality on bounded Euclidean John domains, where the weights are nonnegative powers of the distance to the boundary. In this theorem, we also provide an estimate of the constant involved in the inequality which depends on the exponent that appears in the weight and a geometric condition that characterizes John domains. The proof uses a local-to-global argument based on a certain decomposition of functions.

In addition, we prove the solvability in weighted Sobolev spaces of $\operatorname{div} \mathbf{u} = f$ on the same class of domains. In this case, the weights are nonpositive powers of the distance to the boundary. The constant appearing in this problem is also estimated.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$, and $1 < p < \infty$. The classical Korn inequality states that

$$\|D\mathbf{u}\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega)^{n\times n}} \tag{1.1}$$

for any vector field $\mathbf{u}$ in the Sobolev space $W^{1,p}(\Omega)^n$ under appropriate conditions. By $D\mathbf{u}$ we denote the differential matrix of $\mathbf{u}$ and by $\varepsilon(\mathbf{u})$ its symmetric part,

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Naturally, the constant C depends only on Ω and p. This inequality plays a fundamental role in the analysis of the linear elasticity equations, where $\mathbf{u}$ represents a displacement field of an elastic body. The tensor $\varepsilon(\mathbf{u})$ is called the linearized strain tensor and (1.1) implies the coercivity of the bilinear form associated to the underlying linear equations. The two conditions on the vector field considered by Korn in his seminal works [K1, K2] were: $\mathbf{u} = 0$ on $\partial\Omega$ (usually called *the first case*), and $\int_\Omega \big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\big) = 0$ (*the*

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second case). These two conditions exclude the nonconstant infinitesimal rigid motions (i.e. fields $\mathbf{u}$ such that the right-hand side of (1.1) vanishes while the left-hand side does not).

Inequality (1.1) in the first case can be easily proved on any arbitrary domain Ω by using the divergence theorem [F, H]. Moreover, it is known that the optimal constant is $\sqrt{2}$. In this work, we deal with the Korn inequality in the second case, where its validity depends on the geometry of the domain. This inequality has been studied under different assumptions on the domain. For example, it is known that the inequality is valid if Ω is a star-shaped domain with respect to a ball [R]. This class contains the convex domains. The proof in [R] is based on certain integral representations of the vector field $\mathbf{u}$ in terms of $\varepsilon(\mathbf{u})$. Other authors have studied this inequality on these domains using different arguments [H, KO, T]. Uniform domains also satisfy the Korn inequality. This was proved in [DM] by modifying the extension operator given by Peter Jones [Jo].

The largest known family of domains where (1.1) holds is the class of John domains. This class was introduced by Fritz John [Joh] and named after him by Martio and Sarvas [MS]. Let us recall the definition of this family. A bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is called a *John domain* with parameter $C_J > 1$ if there exists a point $x_0 \in \Omega$ such that every $y \in \Omega$ has a rectifiable curve parameterized by arc length $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = y$, $\gamma(l) = x_0$ and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq \frac{1}{C_J} t \tag{1.2}$$

for all $t \in [0, l]$, where l is the length of γ. Any Lipschitz domain is a John domain. Another example is the Koch snowflake which has a fractal boundary. A version of the Korn inequality different from (1.1) on John domains can be found in [ADM], where it is obtained as a consequence of the main result of that article on the solvability of $\operatorname{div} \mathbf{u} = f$ with an appropriate a priori estimate. Diening et al. [DRS] proved (1.1) on John domains where the vector fields belong to a weighted Sobolev space with weights in the Muckenhoupt class A_p. More recently, a weighted version of the Korn inequality different from the one treated in this article has been shown in [JK], where the weights are also nonnegative powers of the distance to the boundary. The proof is based on a certain improved version of the Poincaré inequality of [Hu], later generalized in [ChW2].

In this note, we are particularly interested in finding an estimate of the constant that appears in the inequality. This problem has been addressed in several articles. For instance, Durán [D] estimates the constant in (1.1), with $p = 2$, in terms of the ratio between the diameter of Ω and that of B when Ω is a star-shaped domain with respect to a ball B. Another recent article [CD] dealing with the estimation of the constant in the Korn inequality considers star-shaped planar domains. This problem has also been studied in the classical reference [HP]. However, we could not find in the literature estimates of the constant in the Korn inequality when Ω is a John domain.

Our main theorem is a weighted version of the Korn inequality on John domains, where the weight is a nonnegative power of the distance to the boundary. Moreover, we estimate the Korn constant in terms of the geometric condition introduced in (5.2). Similar estimates for weighted Poincaré inequalities which depend on the eccentricity of a convex domain has been proved in [ChD, ChW1]; the authors also consider nonnegative powers of the distance to the boundary.

Given a vector field $\mathbf{u}$ we denote by $\eta(\mathbf{u})$ the skew-symmetric part of the differential matrix $D\mathbf{u}$,

$$\eta_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

We denote by $L^p(\Omega, \nu)$ and $W^{1,p}(\Omega, \nu)$ the weighted L^p and Sobolev spaces with measure $\nu(x)\mathrm{d}x$.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain with $n \geq 2$, and let $1 < p < \infty$ and $\beta \in \mathbb{R}_{\geq 0}$. Then there exists a constant C, depending only on n, p and β, such that*

$$\left(\int_\Omega |D\mathbf{u}|^p \rho^{p\beta}\, dx\right)^{1/p} \leq CK^{n+\beta}\left(\int_\Omega |\varepsilon(\mathbf{u})|^p \rho^{p\beta}\, dx\right)^{1/p} \tag{1.3}$$

for all vector fields $\mathbf{u} \in W^{1,p}(\Omega, \rho^{p\beta})^n$ that satisfy $\int_\Omega \eta_{ij}(\mathbf{u})\rho^{\beta p} = 0$ for $1 \leq i < j \leq n$. The function $\rho(x)$ is the distance to the boundary of Ω and the constant K is introduced in the geometric condition (5.2) *below.*

Notice that $\rho^{p\beta}$ does not belong to the A_p Muckenhoupt class for large $\beta > 0$. Thus, many of the techniques that use the theory of singular integral operators and depend on the continuity of the Hardy–Littlewood maximal operator may not be applicable in this case.

The paper is organized as follows: In Section 2, we introduce some definitions and notation. In Section 3, we show how certain decompositions of functions can be used to extend the local validity of the Korn inequality to the whole domain Ω. In this part of the article, Ω is an arbitrary bounded domain. Section 4 deals with the existence of the required decomposition of functions. In Section 5, we apply the results proved in the previous two sections on John domains to demonstrate the main result of the article.

2. Definitions and preliminaries. Throughout the paper, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$, $1 < p, q < \infty$ with $1/p + 1/q = 1$, and $\omega : \Omega \to \mathbb{R}$ is a positive measurable function such that ω^p is integrable over Ω. By $L^p(\Omega, \omega^p)$ we denote the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ equipped with the norm

$$\|u\|_{L^p(\Omega,\omega^p)} := \Big(\int_\Omega |u(x)|^p \omega^p(x)\, dx\Big)^{1/p}.$$

Similarly, we define the weighted Sobolev space $W^{1,p}(\Omega, \omega^p)$ as the space of weakly differentiable functions $u : \Omega \to \mathbb{R}$ with the norm

$$\|u\|_{W^{1,p}(\Omega,\omega^p)} := \bigg(\int_\Omega |u(x)|^p \omega^p(x)\, dx + \sum_{i=1}^{n} \int_\Omega \bigg|\frac{\partial u(x)}{\partial x_i}\bigg|^p \omega^p(x)\, dx\bigg)^{1/p}.$$

Finally, $\omega^p(U) := \int_U \omega^p$.

In what follows, C denotes various positive constants which may vary from line to line. We use C_a or $C(a)$ to denote a constant which only depends on a.

Let us introduce the decompositions considered in this article.

Definition 2.1. Given $m \in \mathbb{N}_0$, let $\mathcal{P}_m$ be the space of polynomials of degree no greater than m. Moreover, let $\{\Omega_t\}_{t\in\Gamma}$ be a collection of open sets that satisfies $\Omega = \bigcup_{t\in\Gamma} \Omega_t$. Now, given $g \in L^1(\Omega)$ such that $\int g\varphi = 0$ for all $\varphi \in \mathcal{P}_m$, we say that a collection $\{g_t\}_{t\in\Gamma}$ of functions is a $\mathcal{P}_m$-*orthogonal decomposition* of g subordinate to $\{\Omega_t\}_{t\in\Gamma}$ if the following three properties are satisfied:

(1) $g = \sum_{t\in\Gamma} g_t$,
(2) $\operatorname{supp}(g_t) \subset \Omega_t$,
(3) $\int_{\Omega_t} g_t\varphi = 0$ for all $\varphi \in \mathcal{P}_m$.

We may also refer to this collection of functions as a $\mathcal{P}_m$-*decomposition*.

Remark. In this paper, we only use $\mathcal{P}_0$-decompositions but we provide the more general definition due to its interest is other applications. Indeed, the version of the Korn inequality treated in this note can be thought of as an estimate of a certain weighted distance to the vector space $\{D\mathbf{w} : \varepsilon(\mathbf{w}) = 0\}$. The reason why $\mathcal{P}_0$-decompositions are sufficient in this case is that any matrix in this vector space has constant coefficients. When more complex vector spaces are involved, such as the ones appearing in the study of the traced-free version of the Korn inequality [Da, R] or in interpolation in Sobolev spaces by polynomials, it would be necessary to work with $\mathcal{P}_m$-decompositions or even more general decompositions with respect to an appropriate vector space $\mathcal{V}$.

A *covering* $\{\Omega_t\}_{t\in\Gamma}$ of Ω is a countable collection of subdomains of Ω that satisfies $\bigcup_t \Omega_t = \Omega$ and the overlap estimate:

$$\sum_{t\in\Gamma} \chi_{\Omega_t}(x) \leq N \quad \text{for all } x \in \Omega. \tag{2.1}$$

This condition is essential in this note, specifically in Sections 3 and 5.

3. A decomposition and weighted Korn inequalities. In this section, we will show that the validity of a weighted version of the Korn inequality on Ω (introduced below) can be obtained from the local validity of this inequality if we have an appropriate decomposition of functions in $L^q(\Omega, \omega^{-q})$. No additional assumptions on the domain are required in this section apart from being bounded.

Given $U \subseteq \Omega$, we say that *weighted Korn inequality*, with weight ω^p, holds on U if

$$\|D\mathbf{u}\|_{L^p(U,\omega^p)} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(U,\omega^p)} \tag{3.1}$$

for any vector field $\mathbf{u} \in W^{1,p}(U, \omega^p)^n$ that satisfies $\int_U \eta_{ij}(\mathbf{u})\omega^p = 0$ for any $1 \leq i < j \leq n$. An equivalent version of (3.1)

$$\inf_{\varepsilon(\mathbf{w})=0} \|D(\mathbf{v} - \mathbf{w})\|_{L^p(U,\omega^p)} \leq C\|\varepsilon(\mathbf{v})\|_{L^p(U,\omega^p)}, \tag{3.2}$$

where $\mathbf{v}$ is an arbitrary vector field in $W^{1,p}(U, \omega^p)^n$. Let us mention that the vector fields that satisfy $\varepsilon(\mathbf{w}) = 0$ are characterized by

$$\mathbf{w}(x) = Ax + b,$$

where $A \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix and $b \in \mathbb{R}^n$.

The integrability of ω^p required in Section 2 will be used several times, in particular to show that

$$L^q(\Omega, \omega^{-q}) \subset L^1(\Omega).$$

Now, given $m \in \mathbb{N}_0$, we denote by $V_m(\Omega, \omega^{-q})$ (or simply V_m) the subspace of $L^q(\Omega, \omega^{-q})$ given by

$$V_m := \Big\{ g \in L^q(\Omega, \omega^{-q}) : \int g\varphi = 0 \text{ for all } \varphi \in \mathcal{P}_m, \text{ and } \\ \text{supp}(g) \text{ intersects a finite number of } \Omega_t\text{'s} \Big\}.$$

Recall that Ω is bounded, so $\mathcal{P}_m \subset L^\infty(\Omega)$. Since $L^q(\Omega, \omega^{-q}) \subset L^1(\Omega)$, we see that V_m is well-defined.

Lemma 3.1. *Given $m \in \mathbb{N}_0$ and a covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω such that each Ω_t intersects a finite number of Ω_s's, the subspace $S_m \subset L^q(\Omega, \omega^{-q})$ defined by*

$$S_m := \{g + \omega^p\psi : g \in V_m \text{ and } \psi \in \mathcal{P}_m\}$$

is dense in $L^q(\Omega,\omega^{-q})$*. Moreover,* $\|g\|_{L^q(\Omega,\omega^{-q})} \leq C\|g+\omega^p\psi\|_{L^q(\Omega,\omega^{-q})}$*, where* C *does not depend on* g *or* ψ*. In the particular case when* $m=0$ *the constant in the previous inequality is equal to* 2*.*

Proof. Let us remark that $\omega^p\psi$ belongs to $L^q(\Omega,\omega^{-q})$. Indeed,

$$\|\omega^p\psi\|^q_{L^q(\Omega,\omega^{-q})} = \int_\Omega \psi^q\omega^{pq}\omega^{-q} \leq \|\psi^q\|_{L^\infty(\Omega)}\|\omega^p\|_{L^1(\Omega)},$$

thus S_m is a subspace of $L^q(\Omega,\omega^{-q})$.

Notice that any function F in $L^q(\Omega,\omega^{-q})$ can be written as $F=h_F+\omega^p\psi_F$, where $h_F\in L^q(\Omega,\omega^{-q})$ with $\int_\Omega h_F\varphi=0$ for all $\varphi\in\mathcal{P}_m$, and ψ_F belongs to $\mathcal{P}_m$. This follows from $\mathcal{P}_m$ being a finite-dimensional vector space. Thus, the proof is basically reduced to showing existence of an approximation of h_F in V_m (the support of h_F does not necessarily intersect a finite collection of Ω_t's). Let us go back to the proof of the existence of the representation of functions in $L^q(\Omega,\omega^{-q})$ mentioned above. Take an orthonormal basis $\{\psi_i\}_{1\leq i\leq M}$ of $\mathcal{P}_m$, where M is the dimension of $\mathcal{P}_m$, with respect to the inner product

$$\langle\psi,\varphi\rangle_\Omega = \int_\Omega \psi(x)\varphi(x)\omega^p(x)\,dx.$$

The basis satisfies $\int_\Omega \psi_i\psi_j\omega^p=\delta_{ij}$, where δ_{ij} is the Kronecker symbol. Thus, we set $h_F:=F-\omega^p\psi_F$ and

$$\psi_F := \sum_{j=1}^M \alpha_{F,j}\psi_j, \tag{3.3}$$

where $\alpha_{F,j} := \int_\Omega F\psi_j$ for any $1\leq j\leq M$. These coefficients are well-defined and

$$|\alpha_{F,j}| \leq \|F\|_{L^q(\Omega,\omega^{-q})}\|\psi_j\|_{L^p(\Omega,\omega^p)}$$

for all j. In addition, using (3.3) we have

$$\begin{aligned} \|h_F\|_{L^q(\Omega,\omega^{-q})} &\\ &\leq \Big(1+\sum_{j=1}^M \|\psi_j\|_{L^p(\Omega,\omega^p)}\|\omega^p\psi_j\|_{L^q(\Omega,\omega^{-q})}\Big)\|F\|_{L^q(\Omega,\omega^{-q})}. \end{aligned} \tag{3.4}$$

Now, to approximate h_F by a function in V_m we will need another orthonormal basis. Specifically, let us take a cube $Q\subset\Omega$ that intersects a finite number of subdomains in $\{\Omega_t\}_{t\in\Gamma}$ and an orthonormal basis $\{\tilde\psi_i\}_{1\leq i\leq M}$ of $\mathcal{P}_m$ with respect to the inner product

$$\langle\psi,\varphi\rangle_Q = \int_Q \psi(x)\varphi(x)\omega^p(x)\,dx.$$

Notice that in this case we use Q instead of Ω, but each $\tilde{\psi}_j$ is a polynomial in $\mathcal{P}_m$ and $\int_\Omega h_F\tilde{\psi}_j$ is still zero for all j. Now, given $\epsilon > 0$, and using the fact that Γ is countable and each Ω_t intersects a finite number of Ω_s's, let $\Omega_\epsilon \subset \Omega$ be an open set that contains Q, intersects a finite number of Ω_t's and

$$\|(1-\chi_{\Omega_\epsilon})h_F\|_{L^q(\Omega,\omega^{-q})} < \epsilon.$$

Then we define $G := g + \omega^p\psi$ with $\psi := \psi_F$ and

$$g(x) := \chi_{\Omega_\epsilon}(x)h_F(x) + \sum_{i=1}^M \chi_Q(x)\omega^p(x)\tilde{\psi}_i(x) \int_{\Omega\setminus\Omega_\epsilon} h_F(y)\tilde{\psi}_i(y)\,dy.$$

Observe that $\operatorname{supp}(g)$ intersects a finite number of Ω_t's, and $\int_\Omega g\tilde{\psi}_j = 0$ for all j, thus $g \in V_m$. Moreover,

$$\begin{aligned}
\|F-G\|_{L^q(\Omega,\omega^{-q})} &= \|h_F - g\|_{L^q(\Omega,\omega^{-q})} \\
&\le \epsilon + \sum_{i=1}^M \Big\|\chi_Q(x)\omega^p(x)\tilde{\psi}_i(x)\int_{\Omega\setminus\Omega_\epsilon} h_F(y)\tilde{\psi}_i(y)\,dy\Big\|_{L^q(\Omega,\omega^{-q})} \\
&\le \epsilon + \sum_{i=1}^M \int_{\Omega\setminus\Omega_\epsilon} |h_F(y)\tilde{\psi}_i(y)|\,dy\, \|\tilde{\psi}_i\omega^p\|_{L^q(Q,\omega^{-q})} \\
&\le \epsilon\Big(1 + \sum_{i=1}^M \|\tilde{\psi}_i\|_{L^p(\Omega,\omega^p)}\|\tilde{\psi}_i\omega^p\|_{L^q(Q,\omega^{-q})}\Big).
\end{aligned}$$

Finally, we only have to estimate the norm of g by the norm of $G = g + \omega^p\psi$. This representation is unique so we can assume that $g = h_G$ and $\psi = \psi_G$ defined above. Thus, from (3.4) we have

$$\begin{aligned}
&\|g\|_{L^q(\Omega,\omega^{-q})} \\
&\quad\le \Big(1 + \sum_{j=1}^M \|\psi_j\|_{L^p(\Omega,\omega^p)}\|\omega^p\psi_j\|_{L^q(\Omega,\omega^{-q})}\Big)\|g+\omega^p\psi\|_{L^q(\Omega,\omega^{-q})}.
\end{aligned}$$

When $m = 0$, the space $\mathcal{P}_0$ has dimension 1 and we take the basis given by the vector

$$\psi_0(x) := \frac{1}{(\omega^p(\Omega))^{1/2}},$$

where $\omega^p(\Omega) := \int_\Omega \omega^p$. Thus, $\|\psi_0\|_{L^p(\Omega,\omega^p)}\|\omega^p\psi_0\|_{L^q(\Omega,\omega^{-q})} = 1$, which yields the constant 2. ∎

The following is the main result of the section.

Theorem 3.2. *If the weighted Korn inequality* (3.2) *is valid on Ω_t, with a uniform constant C_1 for all $t \in \Gamma$, and there exists a $\mathcal{P}_0$-orthogonal decom-*

position of any $g \in V_0(\Omega, \omega^{-q})$ *subordinate to* $\{\Omega_t\}_{t\in\Gamma}$, *with the estimate*

$$\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\omega^{-q})} \le C_0^q \|g\|^q_{L^q(\Omega,\omega^{-q})},$$

then the weighted Korn inequality (3.1) *is valid in* Ω: *there exists a constant* C *such that*

$$\|D\mathbf{u}\|_{L^p(\Omega,\omega^p)} \le C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)} \tag{3.5}$$

for any $\mathbf{u} \in W^{1,p}(\Omega, \omega^p)^n$ *with* $\int_\Omega \eta_{ij}(\mathbf{u})\omega^p = 0$ *for* $1 \le i < j \le n$.

Proof. The differential matrix of $\mathbf{u}$ can be written as the sum of its symmetric part $\varepsilon(\mathbf{u})$ and its skew-symmetric part $\eta(\mathbf{u})$. Thus, in order to prove the theorem, it is necessary and sufficient to show that $\|\eta_{ij}(\mathbf{u})\|_{L^p(\Omega,\omega^p)} \le C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)}$ for $1 \le i < j \le n$.

Now, given $t \in \Gamma$, we have

$$\inf_{\alpha\in\mathcal{P}_0} \|\eta_{ij}(\mathbf{u}) - \alpha\|_{L^p(\Omega_t,\omega^p)} \le C_1\|\varepsilon(\mathbf{u})\|_{L^p(\Omega_t,\omega^p)} \tag{3.6}$$

for any $1 \le i < j \le n$, where C_1 is independent of t.

Let $g + \omega^p\psi$ be an arbitrary function in S_0, with $\|g + \omega^p\psi\|_{L^q(\Omega,\omega^{-q})} \le 1$. The function ψ is simply a constant. Thus, using $\int_\Omega \eta_{ij}(\mathbf{u})\,\omega^p = 0$ and the existence of the $\mathcal{P}_0$-orthogonal decomposition we have

$$\begin{aligned}\int_\Omega \eta_{ij}(\mathbf{u})(g + \omega^p\psi) &= \int_\Omega \eta_{ij}(\mathbf{u})g = \int_\Omega \eta_{ij}(\mathbf{u}) \sum_{t\in\Gamma} g_t \\ &= \sum_{t\in\Gamma} \int_{\Omega_t} \eta_{ij}(\mathbf{u})g_t = \sum_{t\in\Gamma} \int_{\Omega_t} (\eta_{ij}(\mathbf{u}) - \alpha)g_t = (I),\end{aligned}$$

for any $\alpha \in \mathcal{P}_0$. Observe that the sum in the previous lines is finite as g is a function in V_0. Next, applying the Hölder inequality to (I), inequality (3.6) on each Ω_t, and finally the Hölder inequality for the sum, we obtain

$$\begin{aligned}(I) &\le \sum_{t\in\Gamma} \inf_{\alpha\in\mathcal{P}_0} \|(\eta_{ij}(\mathbf{u}) - \alpha)\|_{L^p(\Omega_t,\omega^p)} \|g_t\|_{L^q(\Omega_t,\omega^{-q})} \\ &\le \sum_{t\in\Gamma} C_1\|\varepsilon(\mathbf{u})\|_{L^p(\Omega_t,\omega^p)} \|g_t\|_{L^q(\Omega_t,\omega^{-q})} \\ &\le C_1\Big(\sum_{t\in\Gamma} \int_{\Omega_t} |\varepsilon(\mathbf{u})|^p\omega^p\Big)^{1/p} \Big(\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\omega^{-q})}\Big)^{1/q} = (II).\end{aligned}$$

Now, we use the estimate in the statement of the theorem, the estimate of the overlap of $\{\Omega_t\}_{t\in\Gamma}$ and the estimate of the constant in Lemma 3.1:

$$\begin{aligned}(II) &\le C_1 N^{1/p} C_0 \|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)} \|g\|_{L^q(\Omega,\omega^{-q})} \\ &\le 2C_1 N^{1/p} C_0 \|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)}.\end{aligned}$$

Finally, since S_0 is dense in $L^q(\Omega,\omega^{-q})$, taking the supremum over all $g+\omega^p\psi\in S_0$ with $\|g+\omega^p\psi\|_{L^q(\Omega,\omega^{-q})}\le 1$ we conclude that

$$\|\eta_{ij}(\mathbf{u})\|_{L^p(\Omega,\omega^p)}=\sup_{g+\omega^p\psi}\int_\Omega \eta_{ij}(\mathbf{u})(g+\omega^p\psi)\le 2N^{1/p}C_0C_1\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)}.$$

Thus,

$$\begin{aligned}\|D\mathbf{u}\|_{L^p(\Omega,\omega^p)}&\le\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)}+\|\eta(\mathbf{u})\|_{L^p(\Omega,\omega^p)}\\&\le(1+2n^{2/p}N^{1/p}C_0C_1)\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega^p)},\end{aligned}$$

completing the proof. ∎

Remark 3.3. Notice that the proof of Theorem 3.2 also gives an explicit constant for the weighted Korn inequality (3.5) on Ω. Indeed, we can take

$$C=1+2n^{2/p}N^{1/p}C_0C_1,$$

where C_1 is a uniform constant for inequality (3.5) on each subdomain Ω_t, C_0 is the constant involved in the estimate of the $\mathcal{P}_0$-decomposition and N controls the overlap.

4. A $\mathcal{P}_0$-decomposition on general domains. In this section, we show the existence of a $\mathcal{P}_0$-decomposition subordinate to a covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω if we have a certain order on Γ. The construction follows the ideas in [L], where this kind of technique was used to prove the solvability in weighted Sobolev spaces of the equation $\operatorname{div}\mathbf{u}=f$ on some irregular domains.

Let us denote by $G=(V,E)$ a graph with vertices V and edges E. Graphs in this notes have neither multiple edges nor loops and the number of vertices in V is at most countable. A *rooted tree* (or simply a tree) is a connected graph G in which any two vertices are connected by exactly one simple path, and a *root* is simply a distinguished vertex $a\in V$. The set of vertices V of a tree will be usually denoted by Γ and we may say that Γ has a rooted tree structure without specifying the set of edges E. Moreover, if $G=(\Gamma,E)$ is a rooted tree, it is possible to define a *partial order* "$\preceq$" in Γ as follows: $s\preceq t$ if and only if the unique path connecting t to the root a passes through s. The *height* or *level* of any $t\in\Gamma$ is the number of vertices in $\{s\in\Gamma: s\preceq t \text{ with } s\neq t\}$. *The parent* of a vertex $t\in\Gamma$ is the vertex s satisfying $s\preceq t$ and whose height is 1 smaller than the height of t. We denote the parent of t by t_p. It can be seen that each $t\in\Gamma$ different from the root has a unique parent, but several elements on Γ could have the same parent. Note that two vertices are connected by an edge (*adjacent vertices*) if one is the parent of the other one.

Definition 4.1. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $\{\Omega_t\}_{t\in\Gamma}$ a covering of Ω. We say that $\{\Omega_t\}_{t\in\Gamma}$ is a *tree covering* of Ω if Γ is the set of vertices of a rooted tree, with root $a\in\Gamma$, such that for any $t\in\Gamma$ with

$t \neq a$, there exists an open cube $B_t \subseteq \Omega_t \cap \Omega_{t_p}$ such that the collection $\{B_t\}_{t\neq a}$ is pairwise disjoint.

The tree structure on Γ gives a certain notion of geometry to Ω. We are interested in graph structures which are consistent with the geometry that we already have in Ω. In Section 5, we will show the existence of an appropriate tree structure on Γ, where $\{\Omega_t\}_{t\in\Gamma}$ is a dilation of a Whitney decomposition of a John domain Ω. Similar constructions have been developed in [L] for Hölder-α domains and other examples.

Definition 4.2. Given a tree covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω we define *the Hardy type operator* T as follows:

$$Tg(x) := \sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |g|, \tag{4.1}$$

where $W_t = \bigcup_{s\succeq t} \Omega_s$ and χ_t is the characteristic function of B_t for all $t \neq a$. We may refer to W_t as the *shadow* of Ω_t.

The next lemma is a fundamental result that proves the continuity of the operator T. This result was shown in [L, Lemma 3.1].

Lemma 4.3. *The operator $T : L^q(\Omega) \to L^q(\Omega)$ defined in (4.1) is continuous for any $1 < q < \infty$. Moreover, its norm is bounded by*

$$\|T\|_{L^q\to L^q} \leq 2\left(\frac{qN}{q-1}\right)^{1/q}.$$

It is well-known that the Hardy–Littlewood maximal operator plays an important role in the theory of singular integral operators in weighted spaces. This Hardy type operator plays a similar role when we want to define decompositions of functions in weighted spaces. Another article where Hardy operators have been used to prove a weighted version of Korn's inequality is [AO], where the authors deal with certain domains which have an external cusp.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a tree covering $\{\Omega_t\}_{t\in\Gamma}$. Given $g \in L^1(\Omega)$ such that $\int_\Omega g = 0$ and $\operatorname{supp}(g) \cap \Omega_s \neq \emptyset$ for a finite number of $s \in \Gamma$, there exists a $\mathcal{P}_0$-decomposition $\{g_t\}_{t\in\Gamma}$ of g subordinate to $\{\Omega_t\}_{t\in\Gamma}$ (see Definition 2.1).*

Moreover, fix $t \in \Gamma$. If $x \in B_s$ where $s = t$ or $s_p = t$, then

$$|g_t(x)| \leq |g(x)| + \frac{|W_s|}{|B_s|} Tg(x), \tag{4.2}$$

where W_t denotes the shadow of Ω_t; otherwise

$$|g_t(x)| \leq |g(x)|. \tag{4.3}$$

Proof. Let $\{\phi_t\}_{t\in\Gamma}$ be a partition of unity subordinate to $\{\Omega_t\}_{t\in\Gamma}$, that is a collection of smooth functions such that $\sum_{t\in\Gamma}\phi_t = 1$, $0 \le \phi_t \le 1$ and $\mathrm{supp}(\phi_t) \subset \Omega_t$. Thus, g can be decomposed into $g = \sum_{t\in\Gamma} f_t$ with $f_t = g\phi_t$. This decomposition satisfies (1) and (2) in Definition 2.1 but not necessarily (3). Thus, we will make some modifications to obtain orthogonality to $\mathcal{P}_0$.

Define

$$g_t(x) := f_t(x) + \sum_{s:\, s_p = t} h_s(x) - h_t(x), \tag{4.4}$$

where

$$h_s(x) := \frac{\chi_s(x)}{|B_s|} \int\limits_{W_s} \sum_{k\succeq s} f_k. \tag{4.5}$$

Here χ_t is the characteristic function of B_t. The sum in (4.4) is over $s \in \Gamma$ such that t is the parent of s. When t is the root of Γ, (4.4) means

$$g_a(x) = g_a(x) + \sum_{s:\, s_p = a} h_s(x).$$

Note that the functions h_s in (4.5) are well-defined because g is integrable. Moreover, $h_s \not\equiv 0$ only if $f_t \not\equiv 0$ for some $a \preceq s \preceq t$. Thus, $h_s \not\equiv 0$ for a finite number of $s \in \Gamma$. In addition, we have the following immediate properties,

$$\begin{aligned} &\mathrm{supp}(h_s) \subset B_s, \\ &|h_s(x)| \le \frac{|W_s|}{|B_s|}\chi_s(x) Tg(x) \quad \text{for all } x \in \Omega. \end{aligned} \tag{4.6}$$

Using (4.6) we conclude that $|g_t(x)| \le |g(x)| + \frac{|W_s|}{|B_s|} Tg(x)$ for any $x \in B_s$ with $s = t$ or $s_p = t$, and $|g_t(x)| \le |g(x)|$ otherwise, proving (4.2) and (4.3).

Let us continue by showing that $g(x) = \sum_{t\in\Gamma} g_t(x)$ for all x. Take $x \in \Omega \setminus \bigcup_{k\in\Gamma} B_k$. Then $g_t(x) = f_t(x)$ for all $t \in \Gamma$, and

$$\sum_{t\in\Gamma} g_t(x) = \sum_{t\in\Gamma} f_t(x) = g(x).$$

Otherwise, if x belongs to $B_{\tilde{k}}$ for $\tilde{k} \in \Gamma$, it can be observed that $g_t(x) = f_t(x)$ for all t such that $t \ne \tilde{k}$ and $t \ne \tilde{k}_p$. We are using the fact that the cubes B_s are pairwise disjoint. Moreover,

$$g_{\tilde{k}}(x) = f_{\tilde{k}}(x) - h_{\tilde{k}}(x), \qquad g_{\tilde{k}_p}(x) = f_{\tilde{k}_p}(x) + h_{\tilde{k}}(x).$$

Thus, $\sum_{t\in\Gamma} g_t(x) = g(x)$ for all x.

The second property in Definition 2.1 follows by observing that the parent of each s in (4.4) is t, so $B_s \subseteq \Omega_s \cap \Omega_t$. Thus, $\mathrm{supp}(g_t) \subseteq \Omega_t$.

Finally, in order to prove that g_t is orthogonal to $\mathcal{P}_0$ for all $t \in \Gamma$ observe that $k \succeq t$ if and only if $k \succeq s$ with $s_p = t$, or $k = t$. Thus,

$$\int h_t = \int_{W_s} \sum_{k \succeq t} f_k = \int_{\Omega_t} f_t + \sum_{s\colon s_p = t} \int_{W_s} \sum_{k \succeq s} f_k = \int_{\Omega_t} f_t \; + \sum_{s\colon s_p = t} \int h_s.$$

Therefore, $\int g_t = 0$ for all $t \neq a$. Finally, $\int g_a = \int g = 0$. ∎

5. Korn inequalities and more on John domains. Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain. In the first part of this section, in order to use the results stated in Sections 3 and 4, we will show that there exists a tree covering $\{\Omega_t\}_{t \in \Gamma}$ of Ω for which it is possible to estimate the ratio $|W_t|/|B_t|$ for any $t \in \Gamma \setminus \{a\}$. This covering also satisfies (2.1) and each Ω_t intersects a finite number of Ω_s, with s in Γ.

A *Whitney decomposition* of Ω is a collection $\{Q_t\}_{t \in \Gamma}$ of closed dyadic cubes whose interiors are pairwise disjoint, which satisfies

(1) $\Omega = \bigcup_{t \in \Gamma} Q_t$,
(2) $\operatorname{diam}(Q_t) \leq \rho(Q_t, \partial\Omega) \leq 4 \operatorname{diam}(Q_t)$,
(3) $\frac{1}{4} \operatorname{diam}(Q_s) \leq \operatorname{diam}(Q_t) \leq 4 \operatorname{diam}(Q_s)$ if $Q_s \cap Q_t \neq \emptyset$.

Two different cubes Q_s and Q_t with $Q_s \cap Q_t \neq \emptyset$ are called *neighbors*. Notice that two neighbors may have an intersection with dimension less than $n-1$. For instance, they could intersect in a one-point set. We say that Q_s and Q_t are $(n-1)$-*neighbors* if $Q_s \cap Q_t$ is an $n-1$-dimensional face. This kind of covering exists for any proper open set in $\mathbb{R}^n$ (see [S] for details). Moreover, each cube Q_t has no more than 12^n neighbors. And, if we fix $0 < \epsilon < 1/4$ and define Q_t^* as the cube with the same center as Q_t and side length $1+\epsilon$ times the side length of Q_t, then Q_t^* touches Q_s^* if and only if Q_t and Q_s are neighbors. Thus, each expanded cube has no more than 12^n neighbors and $\sum_{t \in \Gamma} \chi_{Q_t^*}(x) \leq 12^n$.

Definition 5.1. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *Boman chain condition* if there exists a Whitney decomposition $\{Q_t\}_{t \in \Gamma}$ of Ω, with a distinguished cube Q_a, and $\lambda > 1$ such that for any cube Q_t with $t \in \Gamma$ there is a chain of cubes pairwise different $Q_{t,0}, Q_{t,1}, \ldots, Q_{t,\kappa}$ such that $Q_{t,0} = Q_t$, $Q_{t,\kappa} = Q_a$ and

$$Q_{t,i} \subseteq \lambda Q_{t,j} \tag{5.1}$$

for all $0 \leq i \leq j \leq \kappa$, where $\kappa = \kappa(t)$.

Moreover, two consecutive cubes $Q_{t,i-1}$ and $Q_{t,i}$ in this chain are $(n-1)$-neighbors.

This kind of condition was first introduced by Boman [Bom]. Later, Buckley et al. [BKL] proved, in a very general context, that the condition introduced by Boman characterizes John domains. The formulation in Definition 5.1 is slightly different from the one in [BKL], as we see that (5.1) is valid for all $0 \le i \le j$, and not just for $i = 0$ as in [BKL]. Thus, to prove that any bounded John domain satisfies this definition we use [DRS, Theorem 3.8].

LEMMA 5.2. *A bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if and only if it satisfies the Boman chain condition in Definition* 5.1.

Proof. Definition 5.1 implies the definition of Boman chain used in [BKL], so the converse statement in this lemma is proved. Let us show the direct statement. Given a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ of Ω and following [DRS], there is a distinguished cube Q_a and $\lambda > 1$ such that for each cube Q_t there is a chain of pairwise different cubes $Q_{t,0}, Q_{t,1}, \dots, Q_{t,\kappa}$ that connects Q_t to Q_a and satisfies (5.1). Let us modify this chain so that two consecutive cubes in the chain are $(n-1)$-neighbors. Suppose that $F := Q_{t,i-1} \cap Q_{t,i}$ has dimension d in $0 \le d \le n-2$. Then we take $n-d-1$ Whitney cubes intersecting F such that two consecutive cubes in the chain $Q_{t,i-1}, Q_1, \dots, Q_{n-d-1}, Q_{t,i}$ are $(n-1)$-neighbors. Moreover, from the third condition in the definition of Whitney decomposition we know that the dilation by a constant C_n of each cube in this list contains the other ones. Thus, repeating this process between two consecutive cubes in $Q_{t,0}, Q_{t,1}, \dots, Q_{t,\kappa_t}$ and replacing λ by $C_n\lambda$ in (5.1), we obtain a Boman chain of Whitney cubes where two consecutive cubes are $(n-1)$-neighbors. The pairwise different condition is easily recovered, in case it is necessary, by removing the cubes in the chain between the repeated cubes. ∎

REMARK. It is well known that if Ω satisfies the Boman chain condition with a distinguished cube Q_a, then we can take as a distinguished cube any cube in the Whitney decomposition. However, the constant λ in (5.1) may vary.

In order to define an appropriate tree covering of Ω, we have to prove that John domains satisfy the new condition stated below which is richer than the Boman chain condition.

DEFINITION 5.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω satisfies the *Boman tree condition* if there exists a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$, where Γ has a rooted tree structure, that satisfies

$$Q_s \subseteq KQ_t \tag{5.2}$$

for any $s, t \in \Gamma$ with $s \succeq t$. Moreover, if two vertices t and s are adjacent in Γ then Q_t and Q_s must be $(n-1)$-neighbors.

LEMMA 5.4. *The Boman chain condition and the Boman tree condition are equivalent.*

The reverse of the equivalence in Lemma 5.4 is obtained by taking Q_a as the distinguished cube, where a is the root of Γ. Thus, given Q_κ with $\kappa \in \Gamma$, we have $Q_s \subseteq KQ_t$ for all $a \preceq t \preceq s \preceq \kappa$. Observe that in this case the chain starts at Q_a, instead of Q_κ as in (5.1), and ends at Q_κ. The other implication is shown in the Appendix and follows some ideas by A. A. Vasil'eva [V].

We can conclude, from Lemmas 5.2 and 5.4, that Definition 5.3 characterizes bounded John domains in $\mathbb{R}^n$. Regarding the constants that appear in (1.2) and (5.2), it is not clear in general what is the relation between them. In order to have a better understanding of the geometric constant K we exhibit two domains in $\mathbb{R}^2$ where the constants C_J and K can be easily estimated. The first example is the rectangle $\Omega_1 = (-m^2, m^2) \times (0,1)$, for $m \gg 1$, where $C_J \approx K \approx m^2$. The second example, Ω_2, is obtained by twisting Ω_1 as shown in Figure 1. In this case, C_J is still comparable to m^2 while $K \approx m$.

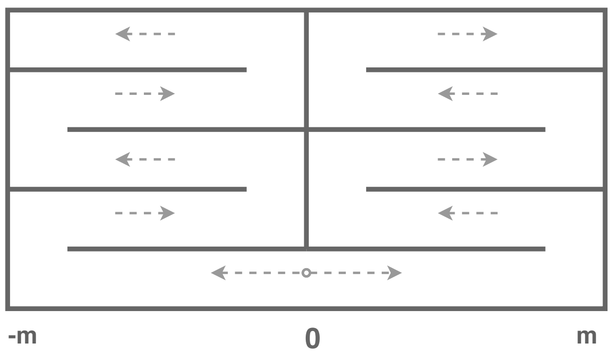

Fig. 1. Twisted rectangle

Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain. Then, from Lemmas 5.2 and 5.4, we know that there exists a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ with all the properties of Definition 5.3. Thus, we define a covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω by

$$\Omega_t := \tfrac{17}{16}Q_t^\circ, \tag{5.3}$$

where $\frac{17}{16}Q_t^\circ$ denotes the open cube with the same center as Q_t and side length $\frac{17}{16}$ times the side length of Q_t.

Corollary 5.5. *The covering $\{\Omega_t\}_{t\in\Gamma}$ of the bounded John domain Ω defined in* (5.3) *is a tree covering with*

$$\operatorname{diam}(\Omega_t) \le C_n \operatorname{diam}(B_t), \tag{5.4}$$

$$\operatorname{diam}\Big(\bigcup_{s\succeq t} \Omega_s\Big) \le K \operatorname{diam}(\Omega_t), \tag{5.5}$$

for any $t \in \Gamma$ ($t \ne a$ in the first inequality), where K is the constant of (5.2).

Moreover, the overlap condition (2.1) *is satisfied with* $N = 12^n$, *each* Ω_t *intersects a finite number of* Ω_s *with* $s \in \Gamma$, *and*

$$\frac{1}{C_n}\operatorname{diam}(\Omega_t) \le \rho(\Omega_t, \partial\Omega) \le C_n \operatorname{diam}(\Omega_t). \tag{5.6}$$

Proof. Regarding (5.5), observe that (5.2) is also valid for the cubes in $\{\Omega_t\}_{t\in\Gamma}$ as we are dilating the cubes in $\{Q_t\}_{t\in\Gamma}$ by the same factor. Thus,

$$\Omega_s \subseteq K\Omega_t$$

for any $t, s \in \Gamma$ with $t \preceq s$, and we obtain (5.5). The rest is a straightforward calculation except the existence of the pairwise disjoint collection $\{B_t\}_{t\neq a}$ satisfying (5.4). We know that Q_t and Q_{t_p} are $(n-1)$-neighbors. Thus, $F_t := Q_t \cap Q_{t_p}$ is an $(n-1)$-dimensional face of the smaller of the two cubes. We denote by α_t the centroid of F_t. Let us use the distance $d_1(x,y) := \max_{1\le i\le n}|x_i - y_i|$, which is more convenient than $d(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$ in this context. Moreover, we use the side length of Q_t, denoted by $l(Q_t)$, instead of $\operatorname{diam}(Q_t)$. Thus, using the third condition in the definition of Whitney decomposition, it can be seen that

$$d_1(\alpha_t, \alpha_s) \ge \tfrac{1}{8} l(Q_t)$$

for all $s \in \Gamma \setminus \{a, t\}$. Thus, if we define B_t as the open cube with center at α_t and side length $l(Q_t)/8$, we obtain a collection of pairwise disjoint cubes. However, it is also required that $B_t \subset \Omega_t \cap \Omega_{t_p}$. Therefore, we take B_t with length side equal to $l(Q_t)/64$ which satisfies the required conditions. Then, (5.4) holds with $C_n = 64$. ■

The next lemma will be used to prove the weighted estimate for the $\mathcal{P}_0$-orthogonal decomposition that appears in Theorem 3.2.

LEMMA 5.6. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded John domain,* $\{\Omega_t\}_{t\in\Gamma}$ *the tree covering defined in* (5.3), *and* $\beta \ge 0$. *Then the operator* T *defined in* (4.1) *and subordinate to* $\{\Omega_t\}_{t\in\Gamma}$ *is continuous from* $L^q(\Omega, \rho^{-q\beta})$ *to itself, where* ρ *is the distance to the boundary of* Ω. *Moreover, its norm is bounded by*

$$\|T\|_{L\to L} \le C_n^\beta \left(\frac{qN}{q-1}\right)^{1/q} K^\beta,$$

where L *denotes* $L^q(\Omega, \rho^{-q\beta})$. *The constant* K *is the one in* (5.2) *and* $N = 12^n$.

It can be seen, after multiplying by an appropriate constant, that the Hardy–Littlewood maximal operator pointwise bounds the Hardy type operator T defined by using the tree covering introduced in (5.3). Thus T is continuous from $L^p(\Omega, \omega^p)$ to itself if ω^p belongs to the A_p Muckenhoupt class. However, negative powers of the distance to $\partial\Omega$ do not belong to this class if the absolute value of the power is sufficiently large. Thus, we have to prove the weighted continuity of T in a different way.

Proof of Lemma 5.6. Given $g \in L^q(\Omega, \rho^{-q\beta})$ we have

$$\int_\Omega |Tg(x)|^q \rho^{-q\beta}(x)\,dx = \int_\Omega \rho^{-q\beta}(x)\bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\int_{W_t} |g(y)|\,dy\bigg|^q dx$$
$$= \int_\Omega \rho^{-q\beta}(x)\bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\int_{W_t} |g(y)|\rho^{-\beta}(y)\rho^{\beta}(y)\,dy\bigg|^q dx =: (1).$$

Now, given $y \in W_t$ there exists $s \succeq t$ such that $y \in \Omega_s$. Thus, it can be seen that

$$\rho(y) \le C_n \operatorname{diam}(\Omega_s) \le C_n K \operatorname{diam}(\Omega_t).$$

Then, using the fact that β is nonnegative we have

$$\rho^\beta(y) \le C_n^\beta K^\beta \operatorname{diam}(\Omega_t)^\beta \le C_n^\beta K^\beta \rho^\beta(x)$$

for all $x \in B_t$. Recall that χ_t is the characteristic function of $B_t \subset \Omega_t$ and $\operatorname{diam}(\Omega_t)$ is comparable to $\rho(\Omega_t, \partial\Omega)$. Thus,

$$(1) \le C_n^{q\beta} K^{q\beta} \int_\Omega \rho^{-q\beta}(x)\bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)\rho^\beta(x)}{|W_t|}\int_{W_t} |g(y)|\rho^{-\beta}(y)\,dy\bigg|^q dx$$
$$= C_n^{q\beta} K^{q\beta} \int_\Omega \bigg|\sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|}\int_{W_t} |g(y)|\rho^{-\beta}(y)\,dy\bigg|^q dx$$
$$= C_n^{q\beta} K^{q\beta} \int_\Omega |T(g\rho^{-\beta})|^q\,dx =: (2).$$

Finally, $g\rho^{-\beta}$ belongs to $L^q(\Omega)$ and T is continuous from $L^q(\Omega)$ to itself (see Lemma 4.3), thus

$$(2) \le C_n^{q\beta} 2^q \frac{qN}{q-1} K^{q\beta} \|g\|^q_{L^q(\Omega,\rho^{-q\beta})}. \blacksquare$$

Proof of Theorem 1.1. Using Theorem 4.4 we conclude that there exists a $\mathcal{P}_0$-decomposition $\{g_t\}_{t\in\Gamma}$ of any integrable function g. This decomposition is subordinate to the tree covering $\{\Omega_t\}_{t\in\Gamma}$ defined in (5.3). Moreover, it satisfies (4.2), which in this case implies that

$$|g_t(x)| \le |g(x)| + C_n K^n Tg(x)$$

for any $x \in \Omega_t$ with $t \in \Gamma$. Thus, by a straightforward calculation we have

$$\int_{\Omega_t} |g_t(x)|^q \rho^{-q\beta}(x)\,dx$$
$$\le 2^{q-1}\Big(\int_{\Omega_t} |g(x)|^q \rho^{-q\beta}(x)\,dx + C_n^q K^{qn} \int_{\Omega_t} |Tg(x)|^q \rho^{-q\beta}(x)\,dx\Big).$$

Next, by using the bound on the overlap and Lemma 5.6, we have the estimate required in Theorem 3.2:

$$\begin{aligned}\sum_{t\in\Gamma}\|g_t\|^q_{L^q(\Omega_t,\rho^{-q\beta})}\\ &\le 2^{q-1}N(\|g\|^q_{L^q(\Omega,\rho^{-q\beta})}+c_n^qK^{qn}\|Tg\|^q_{L^q(\Omega,\rho^{-q\beta})})\\ &\le 2^{q-1}N\left(1+c_n^qK^{qn}\left(C_n^{q\beta}2^q\frac{qN}{q-1}\right)K^{q\beta}\right)\|g\|^q_{L^q(\Omega,\rho^{-q\beta})}.\end{aligned}$$

Moreover, consistently with the notation used in Theorem 3.2, we have

$$C_0=C_{n,p,\beta}K^{n+\beta}.$$

Finally, we show the validity of the Korn inequality (3.2) on Ω_t, with $\omega=\rho^\beta$, with a constant $C_{p,n}$ independent of $t\in\Gamma$. Using the fact that the distance from Ω_t to the boundary of Ω is comparable to $\operatorname{diam}(\Omega_t)$, it is easy to show that the weight is comparable to a constant over Ω_t, indeed,

$$\frac{1}{C_n}\operatorname{diam}(\Omega_t)\le\rho(x)\le C_n\operatorname{diam}(\Omega_t)$$

for all $x\in\Omega_t$. Moreover, the Korn inequality (3.2) with $\omega=1$ is valid on any cube Ω_t with uniform constant. Thus,

$$\begin{aligned}\inf_{\varepsilon(\mathbf{w})=0}\|D(\mathbf{v}-\mathbf{w})\|_{L^p(\Omega_t,\rho^{p\beta})}&\le C_n^\beta\operatorname{diam}(\Omega_t)^\beta\inf_{\varepsilon(\mathbf{w})=0}\|D(\mathbf{v}-\mathbf{w})\|_{L^p(\Omega_t)}\\ &\le C_n^\beta\operatorname{diam}(\Omega_t)^\beta C_{p,n}\|\varepsilon(\mathbf{v})\|_{L^p(\Omega_t)}\\ &\le C_n^\beta C_{p,n}\|\varepsilon(\mathbf{v})\|_{L^p(\Omega_t,\rho^{p\beta})},\end{aligned}$$

with a constant $C_1=C_{p,n,\beta}$. Thus, the validity of (1.3) and the estimate of its constant follow from Theorem 3.2 and Remark 3.3. ∎

5.1. Weighted solutions of divergence problem on John domains. In this subsection, we basically combine [L, Theorem 3.2] and Lemma 5.4 to show the existence of a weighted solution of $\operatorname{div}\mathbf{u}=f$ on John domains. This problem is basic for the theoretical and numerical analysis of the Stokes equations in Ω and has been widely studied (see [G, ADM, Bog, D, DMRT, L] and references therein). The solutions belong to $W_0^{1,q}(\Omega,\rho^{-q\beta})^n$ which is defined as the closure of $C_0^\infty(\Omega)^n$ in the norm

$$\|\mathbf{u}\|_{W_0^{1,q}(\Omega,\rho^{-q\beta})^n}:=\|D\mathbf{u}\|_{L^q(\Omega,\rho^{-q\beta})^{n\times n}}.$$

THEOREM 5.7. *Let $\Omega\subset\mathbb{R}^n$ be a bounded John domain with $n\ge2$, and let $1<q<\infty$ and $\beta\in\mathbb{R}_{\ge0}$. Given $f\in L^q(\Omega,\rho^{-q\beta})$, with $\int_\Omega f=0$, there exists a solution $\mathbf{u}\in W_0^{1,q}(\Omega,\rho^{-q\beta})^n$ of $\operatorname{div}\mathbf{u}=f$ that satisfies*

$$\|D\mathbf{u}\|_{L^q(\Omega,\rho^{-q\beta})}\le C_{n,q,\beta}K^{n+\beta}\|f\|_{L^q(\Omega,\rho^{-q\beta})},$$

where $\rho(x)$ is the distance to the boundary of Ω and K is the constant of (5.2).

Proof. Let us show that the hypotheses **(a)–(f)** of [L, Theorem 3.2] are fulfilled. First, notice that our p and q are swapped in [L]. We also use a different notation for the definition of weighted spaces. Let $\{\Omega_t\}_{t\in\Gamma}$ be a tree covering as in Lemma 5.5. Being a tree covering, it satisfies **(b)**. $\{\Omega_t\}_{t\in\Gamma}$ is obtained by expanding a Whitney decomposition, which implies **(a)** and **(c)**, with $N = 12^n$. Condition **(d)** involves a weight ω which depends on the geometry of Ω:

$$\omega(x) := \begin{cases} |B_t|/|W_t| & \text{if } x \in B_t \text{ for some } t \in \Gamma,\ t \neq a, \\ 1 & \text{otherwise.} \end{cases}$$

Now, from (5.5) it follows that $\omega(x) \geq 1/(C_n K^n)$ for any $x \in \Omega$. Thus, by taking $\bar{\omega} := 1$ and $M_1 := C_n K^n$ we have **(d)**. In order to prove **(e)** we define $\hat{\omega} := \rho^{-\beta}$, and use the fact that ρ is comparable to $\operatorname{diam}(\Omega_t)$ over Ω_t (see (5.6)). Thus, as there are solutions for the divergence problem on cubes with uniform constant, it follows that given $t \in \Gamma$ and $g \in L^q(\Omega_t, \rho^{-q\beta})$, with vanishing mean value, there exists a solution $\mathbf{v} \in W_0^{1,q}(\Omega_t, \rho^{-q\beta})^n$ of $\operatorname{div} \mathbf{v} = g$ with

$$\|D\mathbf{v}\|_{L^q(\Omega_t,\rho^{-q\beta})} \leq C_{n,\beta} \|g\|_{L^q(\Omega_t,\rho^{-q\beta})}.$$

Thus, M_2 is a constant that depends only on n and β.

Finally, **(f)** follows from Lemma 5.6 with $M_T = C_{n,q,\beta} K^\beta$.

The estimate of the constant follows from [L, Theorem 3.2]. ∎

Appendix A. Boman chain implies Boman tree condition. This section is devoted to proving Lemma 5.4.

According to the previous section, $\{Q_t\}_{t\in\Gamma}$ denotes a Whitney decomposition of a bounded domain $\Omega \subset \mathbb{R}^n$ that satisfies the Boman condition (5.1). The center cube Q_a can be arbitrarily chosen. Thus, we take one with the biggest size. Moreover, without loss of generality and in order to simplify the notation we are going to assume that its side length is 1. For any $s \in \Gamma$, we denote by l_s the side length of Q_s. In addition, the elements in the covering are dyadic cubes, thus $l_s = 2^{-m_s}$, where m_s is a nonnegative integer, which may also be denoted by $m(Q_s)$. For example, $m(Q_a) = m_a = 0$.

Let $G = (V, E)$ be a connected graph. Given $v, v' \in V$ we define the distance $k(v, v')$ as the minimal $j \in \mathbb{N}_0$ such that there exists a simple path $(v_0, v_1, \ldots, v_j)$ of length j that connects v to v'. Namely, $v_0 = v$, $v_j = v'$, and the vertices v_i and v_{i+1} are adjacent. The function k depends on V and E.

Lemma A.1. *Let $G = (V, E)$ be a connected graph with a distinguished vertex $v_* \in V$. The graph also satisfies $k(v, v_*) \leq k$ for all $v \in V$, where k is a fixed value in $\mathbb{N}$. Then there exists a subgraph $\tilde{G} = (V, \tilde{E})$ with the same* *vertices which is a rooted tree with root v_* such that $\tilde{k}(v, v_*) \le k$ for all $v \in V$, where $\tilde{k}$ is the distance for the new graph $\tilde{G}$.*

Proof. The rooted tree $\tilde{G}$ is obtained by eliminating edges from E by using an inductive argument. Indeed, we are going to define a collection $G_i := (V_i, E_i)$ of subgraphs of G for each $0 \le i \le k$. The set V_i of vertices has the vertices $v \in G$ such that $k(v, v_*) \le i$. If k_i denotes the distance between vertices in G_i, we define E_i inductively so that G_i is a subtree of G_{i+1} and $k(v, v_*) = k_i(v, v_*)$ for all $v \in V_i$.

Thus, we define $V_0 = \{v_*\}$ and $E_0 = \emptyset$. Next, given $1 \le i \le k$, the process consists in taking exactly one edge that joins each vertex in $V_i \setminus V_{i-1}$ to V_{i-1} and eliminating the other edges. ∎

LEMMA A.2. *Let $\{Q_t\}_{t\in\Gamma}$ be a Whitney decomposition of Ω satisfying condition* (5.1). *Then there exists a tree structure in Γ such that for all $t, t' \in \Gamma$ with $t' \succeq t$, we have*

$$k(t, t') \le l_*(m_{t'} - m_t) + k_*, \tag{A.1}$$

where

$$l_* := (1 + \lambda^{2n})(2 + \log_2(\lambda)) + 1$$
$$k_* := (1 + \lambda^{2n})(2 + \log_2(\lambda)) + l_*(1 + \log_2(\lambda)).$$

The constant λ is the one introduced in (5.1). *In addition, if two vertices s and t are adjacent, then Q_s and Q_t must be neighbors.*

Proof. We use an inductive argument. As mentioned before, we are assuming that Q_a has maximal side length 1. Thus, we will define a collection of rooted trees $G_m = (\Gamma_m, E_m)$ for every $m \in \mathbb{N}_{\ge -1}$ such that G_m is a subgraph of G_{m+1} (i.e. $\Gamma_m \subseteq \Gamma_{m+1}$ and $E_m \subseteq E_{m+1}$) and all of them are subgraphs of $G_\Omega = (\Gamma, E_\Omega)$, where two vertices $t, t' \in \Gamma$ are adjacent in G_Ω if and only if Q_t and $Q_{t'}$ are neighbors. Moreover, $\bigcup_m \Gamma_m = \Gamma$.

Our inductive hypothesis is:

(h1) Γ_m contains all the cubes Q_t with $m_t = m$.
(h2) $m_t \le m + 1 + \log_2(\lambda)$ for all $t \in \Gamma_m$.
(h3) If $t, t' \in \Gamma_m \setminus \Gamma_{m-1}$ with $t' \succeq t$, then $k(t, t') \le \lambda^{2n}$.
(h4) Condition (A.1) is satisfied for any $t, t' \in \Gamma_m$.

Let us start by defining G_{-1} which has $\Gamma_{-1} := \{a\}$ and $E_{-1} := \emptyset$. It can be easily checked that G_{-1} is a subtree of G_Ω that satisfies (h1) to (h4). So, suppose we have a collection $G_{-1}, G_0, \ldots, G_{m-1}$ for $m \ge 0$ with all the properties mentioned above. To construct G_m, let us start by taking all $t \in \Gamma \setminus \Gamma_{m-1}$ with $m_t = m$. In case there is no t with these properties we simply define $G_m := G_{m-1}$. Thus, for each of those indices with $m_t = m$ there exists a chain of cubes satisfying (5.1) that connects Q_t to Q_a via adjacent cubes. However, we are going to consider just the first part of this chain which joins Q_t with a cube Q_s with $s \in \mathit{\Gamma}_{m-1}$. This element s is the first one with this property (considering the order in the chain). Denote this portion of the original chain as $Q_{t,1}, \ldots, Q_{t,r}, Q_{t,r+1}$, with $Q_{t,1} = Q_t$ and $Q_{t,r+1} = Q_s$. Thus $Q_{t,r}$ is a cube with $t \notin \mathit{\Gamma}_{m-1}$. The number $r = r(t)$ of cubes is bounded by $r \leq \lambda^{2n}$. To prove this, observe that all the cubes in the chain intersect each other in a set with Lebesgue measure zero, thus

$$\sum_{j=1}^{r} |Q_{t,j}| \leq \lambda^n |Q_{t,r}|.$$

Since $Q_{t,r}$ has $t \notin \mathit{\Gamma}_{m-1}$, using (h1) we know that $m(Q_{t,r}) \geq m$ and the right hand side above satisfies

$$\lambda^n |Q_{t,r}| = \lambda^n 2^{-nm(Q_{t,r})} \leq \lambda^n 2^{-nm}.$$

Now, $Q_t \subseteq \lambda Q_{t,j}$ for all $1 \leq j \leq r$. Then using $m_t = m$ we have

$$r\lambda^{-n} 2^{-nm} \leq \sum_{j=1}^{r} |Q_{t,j}|.$$

Thus, $r \leq \lambda^{2n}$.

Now, we define an auxiliary graph $G = (V, E)$, where the set V has a vertex $v_* \notin \mathit{\Gamma}$. The rest of the vertices are the indices in $\mathit{\Gamma} \setminus \mathit{\Gamma}_m$ of the cubes in $Q_{t,1}, \ldots, Q_{t,r}$ for all Q_t with $m_t = m$. Regarding the set E, we join two vertices in V by an edge if they are the indices of two consecutive cubes in a chain $Q_{t,1}, \ldots, Q_{t,r}$, or one is v_* and the other is the index of the tail cube $Q_{t,r}$ in a chain $Q_{t,1}, \ldots, Q_{t,r}$. Next, using Lemma A.1 we find that removing some edges from G it is possible to obtain a rooted tree $\tilde{G} = (V, \tilde{E})$ with root v_* such that the length of each chain connecting the vertices to v_* does not exceed λ^{2n}. Finally, in order to construct $\mathit{\Gamma}_m$, we cut off the subtrees added to the artificial vertex v_* and add them to $\mathit{\Gamma}_{m-1}$, specifically to the indices of the cubes $Q_{t,r+1}$ in the tail of chain. This procedure defines a rooted tree $G_m = (\mathit{\Gamma}_m, E_m)$ with root a that contains G_{m-1} as a subgraph. Once we have defined $G_m = (\mathit{\Gamma}_m, E_m)$, it remains to prove that G_m satisfies (h1) to (h4).

Property (h1) follows by construction.

Next, to prove (h2) it is sufficient to consider the case of $s \in \mathit{\Gamma}_m \setminus \mathit{\Gamma}_{m-1}$. By construction λQ_s contains a cube Q_t with $m_t = m$. Consequently, $\lambda \operatorname{diam}(Q_s) \geq \operatorname{diam}(Q_t)$, and after straightforward calculations we obtain $m_s \leq m + \log_2(\lambda)$.

Condition (h3) also follows by construction. We only have to show the validity of (h4) in $\mathit{\Gamma}_m$. For this, we use the inductive hypothesis (h1)–(h4) on $\mathit{\Gamma}_{m-1}$, and the already proven (h1)–(h3) on $\mathit{\Gamma}_m$. Now, given $t, t' \in \mathit{\Gamma}_m$ with $t \preceq t'$ we have to show that (A.1) holds. We may assume that $t' \in \mathit{\Gamma}_m \setminus \mathit{\Gamma}_{m-1}$, otherwise (A.1) follows by using the inductive hypothesis. Thus, $m_{t'} \geq m$. We split the proof into two cases, $m_t \geq m$ and $m_t \leq m-1$.

Suppose $m_t \geq m$. If $t \in \Gamma_{m-j}$ for some $0 \leq j \leq m+1$, then $j \leq 1 + \log_2(\lambda)$. Indeed, by (h2),

$$m \leq m_t \leq m - j + 1 + \log_2(\lambda).$$

Moreover, let $m-j \leq i \leq m$. Then, using (h3) we conclude that the number of indices $s \in \Gamma_i \setminus \Gamma_{i-1}$ such that $t \preceq s \preceq t'$ does not exceed $1 + \lambda^{2n}$. Thus,

$$k(t,t') \leq (1+\lambda^{2n})(2+\log_2(\lambda)). \tag{A.2}$$

Now, using (h1) and (h2) we have

$$m_{t'} - m_t \geq m - m_t \geq m - m - 1 - \log_2(\lambda) = -1 - \log_2(\lambda).$$

Thus, by (A.2),

$$k(t,t') \leq k_* - l_*(1+\log_2(\lambda)) \leq l_*(m_{t'} - m_t) + k_*.$$

Now suppose $m_t \leq m-1$. We know that $m_{t'} \geq m$, thus there exist two consecutive vertices t_1, t_2 such that $t \preceq t_1 \prec t_2 \preceq t'$, $m_{t_1} \leq m-1$ and $m_{t_2} \geq m$. Now, t_2 and t' are as in the previous situation, so we use (A.2) to obtain

$$k(t_2,t') \leq (1+\lambda^{2n})(2+\log_2(\lambda)).$$

Note that from (h1) we see that t and t_1 belong to Γ_{m-1}, and the inductive hypothesis yields

$$\begin{aligned} k(t,t') &= k(t,t_1) + k(t_1,t_2) + k(t_2,t') \\ &\leq l_*(m_{t_1} - m_t) + k_* + 1 + (1+\lambda^{2n})(2+\log_2(\lambda)) \\ &= l_*(m_{t_1} - m_t) + k_* + l_* \leq l_*(m_{t'} - m_t) + k_*, \end{aligned}$$

concluding the proof. ∎

Proof of Lemma 5.4. This result is a corollary of Lemma A.2. Indeed, given $s,t \in \Gamma$ with $t \preceq s$, we denote by α_t the center of Q_t and take an arbitrary $y \in Q_s$. Since two adjacent vertices in Γ are the indices of neighbor cubes, we have

$$\begin{aligned} \operatorname{dist}(\alpha_t, y) &\leq \sum_{t \preceq t' \preceq s} \operatorname{diam}(Q_{t'}) = \sqrt{n} \sum_{t \preceq t' \preceq s} 2^{-m_{t'}} \\ &= \sqrt{n}\, 2^{-m_t} \sum_{t \preceq t' \preceq s} 2^{-(m_{t'} - m_t)} =: (I). \end{aligned}$$

Next, from (A.1),

$$(I) \leq \sqrt{n}\, 2^{-m_t} \sum_{t \preceq t' \preceq s} 2^{-\frac{1}{l_*}(k(t,t') - k_*)} = \sqrt{n}\, 2^{-m_t} 2^{k_*/l_*} \sum_{i=0}^{k(t,s)} (2^{-1/l_*})^i.$$

Finally, the following constant fulfills (5.2):

$$K := 2^{1+k_*/l_*}\sqrt{n}\sum_{i=0}^{\infty}(2^{-1/l_*})^i. \ \blacksquare$$

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