A DECOMPOSITION TECHNIQUE FOR INTEGRABLE FUNCTIONS WITH APPLICATIONS TO THE DIVERGENCE PROBLEM

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain that can be written as $\Omega = \bigcup_t \Omega_t$, where $\{\Omega_t\}_{t\in\Gamma}$ is a countable collection of domains with certain properties. In this work, we develop a technique to decompose a function $f \in L^1(\Omega)$, with vanishing mean value, into the sum of a collection of functions $\{f_t - \tilde{f}_t\}_{t\in\Gamma}$ subordinated to $\{\Omega_t\}_{t\in\Gamma}$ such that $Supp\,(f_t - \tilde{f}_t) \subset \Omega_t$ and $\int f_t - \tilde{f}_t = 0$. As an application, we use this decomposition to prove the existence of a solution in weighted Sobolev spaces of the divergence problem $\operatorname{div} \mathbf{u} = f$ and the well-posedness of the Stokes equations on Hölder-α domains and some other domains with an external cusp arbitrarily narrow. We also consider arbitrary bounded domains. The weights used in each case depend on the type of domain.

1. Introduction

In this paper we show a kind of atomic decomposition for an integral function $f \in L^1(\Omega)$ if Ω is a bounded domain which can be written as the union of a countable collection of domains $\{\Omega_t\}_{t\in\Gamma}$ with certain properties. This result is based on a decomposition developed by Bogovskii in [5], where Γ is finite. The goal of this result is to write a function f with $\int f = 0$ as the sum of a collection of functions $\{f_t - \tilde{f}_t\}_{t\in\Gamma}$ such that $Supp\{f_t - \tilde{f}_t\} \subset \Omega_t$ and $\int_{\Omega_t} f_t - \tilde{f}_t = 0$. As Bogovskii did in his paper we use this decomposition to study the existence of solutions of the divergence problem, and posteriorly the well-posedness of the Stokes equations.

Let us introduce the divergence problem for a bounded domain $\Omega \subset \mathbb{R}^n$. Given $f \in L^p(\Omega)$, with vanishing mean value and $1 < p < \infty$, the divergence problem deals with the existence of a solution $\mathbf{u}$ in the Sobolev space $W_0^{1,p}(\Omega)^n$ of $\operatorname{div} \mathbf{u} = f$ satisfying

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C_\Omega \|f\|_{L^p(\Omega)}, \tag{1.1}$$

where $D\mathbf{u}$ is the differential matrix of $\mathbf{u}$. This problem has been widely studied and it has many applications, for example, in the particular case $p = 2$, it is fundamental for the variational analysis of the Stokes equations (see [13]). It is also well known for its relation with some inequalities such as Korn and Sobolev Poincaré.

Consequently, several methods have been developed to prove the existence of a solution of $\operatorname{div} \mathbf{u} = f$ satisfying (1.1) under different assumptions on the domain (see for example [3], [4], [5], [6], [11], [18]).

On the other hand, this result fails if Ω has an external cusp or arbitrarily narrow "corridors", see for example [2] and [12]. However, the existence of solutions of the divergence problem holds in some of these irregular domains if we consider weighted Sobolev spaces with an estimate weaker than (1.1). A similar analysis can be done for its related results. Since the non-existence of standard solutions arises because of the bad behavior of the boundary, it seems natural to work with weights involving the distance to the boundary of Ω or a subset of it. The following are some papers considering the divergence problem or related results in weighted Sobolev spaces [1], [6], [8], [9] and [19].

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Another point of interest is the characterization of the domains where there exists a standard solution of the divergence equation. This problem has been completely solved if Ω is a bounded planar simply connected domain where it was proved that there exists a solution $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ of $\operatorname{div} \mathbf{u} = f$ satisfying (1.1) if and only if Ω is a John domain. The case $1 < p < 2$ was published in [3] while $2 \leq p < \infty$ was recently shown in [14].

As we mentioned before there are many different approaches to this problem. In the present paper, as it was done in [6] and [10], we use a decomposition of the function f in $\operatorname{div} \mathbf{u} = f$ to generalize results valid on simple domains, such as rectangles or star-shaped domains, to more general cases.

The paper is organized in the following way: In Section 2, we include some notations and preliminary results. In Section 3, we show the main result of this paper, a decomposition technique for integrable functions defined over a bounded domain Ω which is written as the union of a collection of subdomains $\{\Omega_t\}_{t\in\Gamma}$ with some properties. The set Γ is required to have a certain partial order structure. In the following sections we include three different applications of the decomposition developed in Section 3. These sections can be independently read. In section 4, we show the existence of a weighted right inverse of the divergence operator on arbitrary bounded domains. In Sections 5 and 6, we prove the existence of a solution of the divergence problem and the well-posedness of the Stokes equations on some domains with an external cusp arbitrarily narrow and on bounded Hölder-α domains in $\mathbb{R}^n$. The weights in these two final sections are more specific than the one used in Section 4. More precisely, the weights are related to the distance to the cusp and to the distance to the boundary of the domain respectively.

2. Preliminaries and notations

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given a measurable positive function $\omega : \Omega \to R_{>0}$ we denote with $L^p(\Omega, \omega)$ the weighted space with norm

$$\|f\|_{L^p(\Omega,\omega)} = \|f\omega\|_{L^p(\Omega)},$$

and with $W_0^{1,p}(\Omega, \omega)$ the weighted Sobolev space defined as the closure of $C_0^\infty(\Omega)$ with norm

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \|Du\|_{L^p(\Omega,\omega)},$$

where Du is the differential matrix of u. Observe that the seminorm $\|Du\|$ is a norm in the trace zero space.

We say that Ω satisfies $(\text{div})_p$, for $1 < p < \infty$, with constant C_Ω if for any $f \in L_0^p(\Omega) := \{g \in L^p(\Omega) \,:\, g \text{ has vanishing mean value }\}$ there is a solution $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ of $\operatorname{div} \mathbf{u} = f$ satisfying (1.1). We also use C_A to denote a constant depending on A, where A is not necessarily a domain.

In the next lemma we compare C_Ω with $C_{\hat{\Omega}}$, where Ω is a domain obtained by applying an affine function to a domain $\hat{\Omega}$ satisfying $(\text{div})_p$. This result is standard and the proof uses the Piola transform. Before stating with the lemma and given an invertible matrix $B \in \mathbb{R}^{n\times n}$, let us recall the conjugate operator $T_B : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $T_B(A) = BAB^{-1}$. Moreover, let us consider its norm

$$\|T_B\| := \sup_{A\neq 0} \frac{\|T_B(A)\|_p}{\|A\|_p},$$

where $\|A\|_p = \left(\sum_{1\leq i,j\leq n} |A_{i,j}|^p\right)^{1/p}$.

Lemma 2.1. *Let $\hat{\Omega} \subset \mathbb{R}^n$ be a domain satisfying $(\text{div})_p$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ an affine function defined by $F(\hat{x}) = B\hat{x} + b$, where $B \in R^{n\times n}$ is an invertible matrix and $b \in \mathbb{R}^n$. Then, $\Omega = F(\hat{\Omega})$ satisfies $(\text{div})_p$ with a constant C_Ω bounded by*

$$C_\Omega \leq \|T_B\| C_{\hat{\Omega}}.$$

In particular, $C_\Omega = C_{\hat\Omega}$ *if* $B = \lambda I$*, where* I *is the identity matrix and* $\lambda \neq 0$ *is a real number.*

Proof. In order to simplify the notation we assume that all the vectors are in $\mathbb{R}^{n\times 1}$. Given $f \in L^p_0(\Omega)$, the function $\hat g(\hat x) = f(F(\hat x))$ belongs to $L^p_0(\hat\Omega)$. Thus, we define the vector field $\mathbf{u}(x) := B\hat{\mathbf{v}}(F^{-1}(x))$, where $\hat{\mathbf{v}} \in W^{1,p}_0(\hat\Omega)^n$ is a solution of $\operatorname{div}\hat{\mathbf{v}} = \hat g$, with

$$\|\hat{\mathbf{v}}\|_{W^{1,p}_0(\hat\Omega)} \leq C_{\hat\Omega}\|\hat g\|_{L^p(\hat\Omega)}.$$

It can be seen that the differential matrix of $\mathbf{u}$ is $D\mathbf{u}(x) = B\, D\hat{\mathbf{v}}(F^{-1}(x))\, B^{-1}$, and as the trace is invariant under conjugation we can assert that $\operatorname{div}\mathbf{u}(x) = \operatorname{div}\hat{\mathbf{v}}(F^{-1}(x)) = f(x)$. On the other hand, using change of variables it can be seen that

$$\begin{aligned}\|D\mathbf{u}\|_{L^p(\Omega)} &\leq \|T_B\|\det(B)^{1/p}\|D\hat{\mathbf{v}}\|_{L^p(\hat\Omega)}\\ &\leq \|T_B\|\det(B)^{1/p}C_{\hat\Omega}\|\hat g\|_{L^p(\hat\Omega)} = \|T_B\|C_{\hat\Omega}\|f\|_{L^p(\Omega)}.\end{aligned}$$

□

As we mentioned before an important application of the existence of a right inverse of the divergence operator is the well-posedness of the Stokes equations, given by:

$$\begin{cases} -\Delta\mathbf{u} + \nabla p &= \mathbf{g} \quad \text{in } \Omega\\ \operatorname{div}\mathbf{u} &= h \quad \text{in } \Omega\\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega.\end{cases} \tag{2.2}$$

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ (or more generally a John domain [3]), it is known that, if $\mathbf{g} \in H^{-1}(\Omega)^n$, the dual space of $H^1_0(\Omega)^n$, and $h \in L^2(\Omega)$ with vanishing mean value, there exists a unique variational solution $(\mathbf{u}, p)$ in $H^1_0(\Omega)^n \times L^2(\Omega)$. Moreover, this solution satisfies the a priori estimate:

$$\|D\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\left(\|\mathbf{g}\|_{H^{-1}(\Omega)} + \|h\|_{L^2(\Omega)}\right),$$

where the constant C depends only on Ω .

On the other hand, it is known that this result fails in general for domains with external cusps. However, it was proved in [9] that the well-posedness of the incompressible Stokes equations ($h = 0$ in (2.2)) is valid in weighted Sobolev spaces for an arbitrary bounded domain Ω if there exists a standard solution of $\operatorname{div}\mathbf{u} = f$, where f is in a weighted Sobolev spaces. This result is stated bellow.

Theorem. *Let* $\omega \in L^1(\Omega)$ *be a positive function. Assume that for any* $f \in L^2(\Omega, \omega^{-1/2})$*, with vanishing mean value, there exists* $\mathbf{u} \in H^1_0(\Omega)^n$ *such that* $\operatorname{div}\mathbf{u} = f$ *and*

$$\|D\mathbf{u}\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega,\omega^{-1/2})},$$

with a constant C *depending only on* Ω *and* ω*. Then, for any* $\mathbf{g} \in H^{-1}(\Omega)^n$*, there exists a unique* $(\mathbf{u}, p) \in H^1_0(\Omega)^n \times L^2(\Omega, \omega^{1/2})$*, with* $\int_\Omega p\omega = 0$*, incompressible solution of the Stokes problem (2.2). Moreover,*

$$\|D\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega,\omega^{1/2})} \leq C\|\mathbf{g}\|_{H^{-1}(\Omega)},$$

where C *depends only on* Ω *and* ω*.*

3. A decomposition technique for integrable functions

We start this section with an example of Bogovskii's decomposition when Ω is a domain written as the union of a collection of subdomains $\{\Omega_i\}_{0\leq i\leq 2}$. We present the example using our notation. Let $f \in L^p(\Omega)$ be a function with vanishing mean value. Thus, using a partition of the unity $\{\phi_i\}_{0\leq i\leq 2}$ subordinated to $\{\Omega_i\}_{0\leq i\leq 2}$ we can write f as:

$$f = f_0 + f_1 + f_2 = f\phi_0 + f\phi_1 + f\phi_2.$$

Now,

$$f = f_0 + \left(f_1 + \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2\right) + \underbrace{\left(f_2 - \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2\right)}_{f_2-\tilde{f}_2},$$

where $B_2 = \Omega_2 \cap \Omega_1$. Note that the function $f_2 - \tilde{f}_2$ has its support in Ω_2 and $\int f_2 - \tilde{f}_2 = 0$. Finally, we repeat the process with the first two functions. Thus, if $B_1 = \Omega_1 \cap \Omega_0$ we have that

$$\begin{aligned} f &= \overbrace{\left(f_0 + \frac{\chi_{B_1}}{|B_1|}\int_{\Omega_1\cup\Omega_2} f_1 + f_2\right)}^{f_0-\tilde{f}_0} \\ &+ \underbrace{\left(f_1 + \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2 - \frac{\chi_{B_1}}{|B_1|}\int_{\Omega_1\cup\Omega_2} f_1 + f_2\right)}_{f_1-\tilde{f}_1} + \underbrace{\left(f_2 - \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2\right)}_{f_2-\tilde{f}_2}, \end{aligned} \tag{3.3}$$

obtaining the claimed decomposition. Note that we have used the vanishing mean value of f only to prove that $f_0 - \tilde{f}_0$ integrates zero. If we do not assume any other thing but integrability, we have that $\int f_i - \tilde{f}_i = 0$ if $i \neq 0$ and $\int f_0 - \tilde{f}_0 = \int f$.

In this work we extend the decomposition shown in (3.3) when Ω is the union of a collection of subdomains $\{\Omega_t\}_{t\in\Gamma}$, where Γ is a partial ordered countable set instead of a totally ordered finite set. In fact, Γ is a rooted tree and the partial order is inherited from the graph structure.

Let us recall some definitions. A **rooted tree** is a connected graph in which any two vertices are connected by exactly one simple path, and a **root** is simply a distinguished vertex $a \in \Gamma$. For these graphs it is possible to define a **partial order** $\preceq$ as $s \preceq t$ if and only if the unique path connecting t with the root a passes through s. Moreover, the **height** or **level** of any $t \in \Gamma$ is the number of vertices in $\{s \in \Gamma : s \preceq t \text{ with } s \neq t\}$. **The parent** of a vertex $t \in \Gamma$ is the vertex s satisfying that $s \preceq t$ and its height is one unit smaller than the height of t. We denote the parent of t by t_p. It can be seen that each $t \in \Gamma$ different from the root has a unique parent, but several elements on Γ could have the same parent.

3.1. **A "tree" of domains.** Our decomposition for functions in $L^1(\Omega)$ is subordinated to a given decomposition of Ω, which has to satisfy the properties stated below. Thus, let $\{\Omega_t\}_{t\in\Gamma}$ be a countable collection of subdomains of Ω, where Γ is a tree with root a, that satisfies the following properties:

- **(a)** $\chi_\Omega(x) \leq \sum_{t\in\Gamma} \chi_{\Omega_t}(x) \leq N\chi_\Omega(x)$, for almost every $x \in \Omega$.
- **(b)** For any $t \in \Gamma$ different from the root there exists a set $B_t \subseteq \Omega_t \cap \Omega_{t_p}$ with no trivial Lebesgue measure. In addition, the collection $\{B_t\}_{t\neq a}$ is pairwise disjoint.

Finally, given $t \in \Gamma$, we define $W_t = \bigcup_{s\succeq t} \Omega_s$ and we denote the characteristic function of B_t by χ_t, if $t \neq a$.

3.2. **A decomposition on a "tree" of domains.** Let $\{\phi_t\}_{t\in\Gamma}$ be a partition of the unity subordinated to $\{\Omega_t\}_{t\in\Gamma}$. Thus, f can be decomposed into $f = \sum_{t\in\Gamma} f_t$, where $f_t = f\phi_t$, and

$$\sum_{t\in\Gamma} \|f_t\|^p_{L^p(\Omega_t)} \leq N\|f\|^p_{L^p(\Omega)}.$$

Thus, similarly to (3.3), we define $\tilde{f}_t$ for $t \in \Gamma$ as

$$\tilde{f}_t(x) := \frac{\chi_t(x)}{|B_t|}\int_{W_t} \sum_{k\succeq t} f_k - \sum_{s:\, s_p=t} \frac{\chi_s(x)}{|B_s|}\int_{W_s} \sum_{k\succeq s} f_k, \tag{3.4}$$

where the second sum is indexed over all the $s \in \Gamma$ such that t is the parent of s. In the particular case when t is the root of Γ, formula (3.4) means

$$\tilde{f}_a(x) = -\sum_{s:\, s_p=a} \frac{\chi_s(x)}{|B_s|} \int_{W_s} \sum_{k \succeq s} f_k.$$

In the next theorem we prove that f can be written as $\sum_{t\in\Gamma} f_t - \tilde{f}_t$, and some properties of this decomposition, but before that let us define an important operator and show its continuity. Let $T : L^p(\Omega) \to L^p(\Omega)$ be the operator defined by

$$Tf(x) := \sum_{a \neq t \in \Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |f|. \tag{3.5}$$

Lemma 3.1. *The operator $T : L^p(\Omega) \to L^p(\Omega)$ defined in (3.5) is weak $(1,1)$ continuous and strong (p,p) continuous for $1 < p \leq \infty$ with*

$$\|T\|_{L^p \to L^p} \leq 2 \left(\frac{pN}{p-1} \right)^{1/p}.$$

Proof. We prove first that T is strong (∞,∞) continuous and weak $(1,1)$ continuous. Then, using an interpolation theorem, we extend the result to all $1 < p < \infty$.

T is an average of f when it is not zero, thus by a straightforward calculation, it can be proved that T is continuous from L^∞ to L^∞, with norm $\|T\|_{L^\infty \to L^\infty} \leq 1$. In order to prove the weak $(1,1)$ continuity, and given $\lambda > 0$, we define the subset of minimal vertices $\Gamma_0 \subseteq \Gamma$ as

$$\Gamma_0 := \left\{ t \in \Gamma \,:\, \frac{1}{|W_t|} \int_{W_t} |f| > \lambda \text{ and } \frac{1}{|W_s|} \int_{W_s} |f| \leq \lambda \text{ for all } s \preceq t \text{ different from } t \right\}.$$

Thus,

$$\begin{aligned} |\{x \in \Omega \,:\, Tf(x) > \lambda\}| &\leq \sum_{t \in \Gamma_0} |W_t| \\ &< \frac{1}{\lambda} \sum_{t \in \Gamma_0} \int_{W_t} |f| \leq \frac{N}{\lambda} \|f\|_{L^1(\Omega)}, \end{aligned}$$

where N was defined in **(a)** on page 4 and it controls the overlapping of the collection $\{\Omega_t\}$. Thus, T is weak $(1,1)$ continuous with norm lesser than or equal to N.

Finally, using Marcinkiewicz interpolation (see Theorem 2.4 on [7]) T is strong (p,p) continuous, and its norm is lesser than $2\left(\dfrac{pN}{p-1}\right)^{1/p}$. □

Now, we define the weight $\omega : \Omega \to \mathbb{R}_+$ by

$$\omega(x) := \begin{cases} \dfrac{|B_t|}{|W_t|} & \text{if } x \in B_t \text{ for some } t \in \Gamma,\ t \neq a \\[2ex] 1 & \text{otherwise.} \end{cases} \tag{3.6}$$

Let us observe that the collection $\{B_t\}_{t\in\Gamma}$ is pairwise disjoint, thus the weight is well defined. Moreover, $0 < \omega(x) \leq 1$ for all $x \in \Omega$.

Theorem 3.1 (Decomposition technique)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for which there exists a decomposition $\{\Omega_t\}_{t\in\Gamma}$ that fulfills* **(a)** *and* **(b)***. Given $f \in L^1(\Omega)$, and $1 < p < \infty$, the* *decomposition $f=\sum_{t\in\Gamma} f_t-\tilde{f}_t$ defined on (3.4) satisfies that $Supp(f_t-\tilde{f}_t)\subset\Omega_t$, $\int_{\Omega_t} f_t-\tilde{f}_t=0$ for all $t\neq a$, $\int_{\Omega_a} f_a-\tilde{f}_a=\int_\Omega f$, and*

$$\sum_{t\in\Gamma}\|f_t-\tilde{f}_t\|^p_{L^p(\Omega_t,\omega)}\le C_1\|f\|^p_{L^p(\Omega)}, \tag{3.7}$$

where $C_1=2^pN\left(1+\dfrac{2^{p+1}p}{p-1}\right)$.

In addition, if $\hat{\omega}_1,\hat{\omega}_2:\Omega\to\mathbb{R}_{>0}$ are two weights satisfying that $L^p(\Omega,\hat{\omega}_2)\subset L^1(\Omega)$, and the identity operator I and T are continuous from $L^p(\Omega,\hat{\omega}_2)$ to $L^p(\Omega,\hat{\omega}_1)$ with norms M_I and M_T, the decomposition mentioned above also satisfies the following estimate

$$\sum_{t\in\Gamma}\|f_t-\tilde{f}_t\|^p_{L^p(\Omega_t,\omega\hat{\omega}_1)}\le C_2\|f\|^p_{L^p(\Omega,\hat{\omega}_2)}, \tag{3.8}$$

where $C_2=2^p(NM_I^p+2M_T^p)$.

Proof. Observe that B_t and all the B_s on the identity (3.4) are included in Ω_t, thus it follows that $Supp(f_t-\tilde{f}_t)\subset\Omega_t$.

Let us remark that $\tilde{f}_t(x)\neq 0$ only if x belongs to B_s for some $s\in\Gamma-\{a\}$ with $t=s$ or $t=s_p$. Moreover, given x in B_s it follows that $\tilde{f}_s(x)+\tilde{f}_{s_p}(x)=0$, concluding that

$$\sum_{t\in\Gamma} f_t(x)-\tilde{f}_t(x)=\sum_{t\in\Gamma} f_t(x)-\sum_{t\in\Gamma}\tilde{f}_t(x)=f(x)+0.$$

On the other hand, in order to prove the vanishing mean value of $f_t-\tilde{f}_t$, with $t\neq a$,

$$\int_{\Omega_t}\tilde{f}_t=\int_{W_t}\sum_{k\succeq t} f_k-\sum_{s:\,s_p=t}\int_{W_s}\sum_{k\succeq s} f_k=\sum_{k\succeq t}\int_\Omega f_k-\sum_{\substack{k\succeq t\\ k\neq t}}\int_\Omega f_k=\int_{\Omega_t} f_t,$$

obtaining that $\int_{\Omega_t} f_t-\tilde{f}_t=0$. The case $t=a$ follows from

$$\int_{\Omega_a} f_a-\tilde{f}_a=\int_{\Omega_a} f_a+\sum_{s:\,s_p=a}\int_{W_s}\sum_{k\succeq s} f_k=\int_\Omega f_a+\sum_{k\neq a}\int_\Omega f_k=\int_\Omega f.$$

Let us continue with the proof of (3.8). Thus,

$$\sum_{t\in\Gamma}\|f_t-\tilde{f}_t\|^p_{L^p(\Omega_t,\omega\hat{\omega}_1)}\quad\le\quad 2^p\sum_{t\in\Gamma}\|f_t\|^p_{L^p(\Omega_t,\omega\hat{\omega}_1)}+2^p\sum_{t\in\Gamma}\|\tilde{f}_t\|^p_{L^p(\Omega_t,\omega\hat{\omega}_1)}=(I')+(II').$$

Using that $0\le\phi_i,\,\omega\le 1$ and the overlapping of the collection Ω_t is not bigger than N, it follows that

$$(I')\le 2^pN\|f\|^p_{L^p(\Omega,\hat{\omega}_1)}\le 2^pNM_I^p\|f\|^p_{L^p(\Omega,\hat{\omega}_2)}.$$

On the other hand, using that the collection $\{B_t\}_{t\neq a}$ is pairwise disjoint, it can be observed for any $t\neq a$ that

$$\begin{aligned}
\Big(|\tilde{f}_t(x)|\omega(x)\hat{\omega}_1(x)\Big)^p &\le \left(\frac{\omega(x)\chi_t(x)}{|B_t|}\int_{W_t}|f|\;+\;\sum_{s:\,s_p=t}\frac{\omega(x)\chi_s(x)}{|B_s|}\int_{W_s}|f|\right)^p\hat{\omega}_1(x)^p\\
&= \left(\frac{\chi_t(x)}{|W_t|}\int_{W_t}|f|\;+\;\sum_{s:\,s_p=t}\frac{\chi_s(x)}{|W_s|}\int_{W_s}|f|\right)^p\hat{\omega}_1(x)^p\\
&= \left(\left(\frac{\chi_t(x)}{|W_t|}\int_{W_t}|f|\right)^p\;+\;\sum_{s:\,s_p=t}\left(\frac{\chi_s(x)}{|W_s|}\int_{W_s}|f|\right)^p\right)\hat{\omega}_1(x)^p.
\end{aligned}$$

The case $t = a$ is analogous. Hence,

$$\begin{aligned}\sum_{t\in\Gamma}\int_{\Omega_t}\left(|\tilde{f}_t(x)|\omega(x)\hat{\omega}_1(x)\right)^p &\leq 2\int_\Omega \sum_{a\neq t\in\Gamma}\left(\frac{\chi_t(x)}{|W_t|}\int_{W_t}|f|\right)^p\hat{\omega}_1(x)^p \\ &= 2\int_\Omega\left(\sum_{a\neq t\in\Gamma}\frac{\chi_t(x)}{|W_t|}\int_{W_t}|f|\right)^p\hat{\omega}_1(x)^p.\end{aligned}$$

Finally,

$$(II') \quad\leq\quad 2^{p+1}\int_\Omega (Tf(x)\hat{\omega}_1(x))^p \leq 2^{p+1}M_T^p\|f\|^p_{L^p(\Omega,\hat{\omega}_2)},$$

ending the proof of (3.8).

Using the continuity of T proved in Lemma 3.1, it can be seen that (3.7) follows from (3.8) □

3.3. **An application: Divergence problem.** In this subsection, we apply Theorem 3.1 to show the existence of a weighted solution of the divergence problem on some bounded domains $\Omega \subset \mathbb{R}^n$. In fact, this result can be applied if there exists a collection $\{\Omega_t\}_{t\in\Gamma}$ of subdomains of Ω verifying **(a)** and **(b)** in Subsection 3.1, and the additional four conditions stated bellow:

- **(c)** For any point $x \in \Omega$ there exists an open set U containing x such that $U \cap \Omega_t \neq \emptyset$ for a finite number of Ω_t's (this finite number does not need to be bounded by N).
- **(d)** There exists a weight $\bar{\omega} : \Omega \to \mathbb{R}_{>0}$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega_t}\bar{\omega}(x) \leq M_1 \operatorname*{ess\,inf}_{x\in\Omega_t}\omega(x),$$

  for all $t \in \Gamma$, where ω is the weight defined in (3.6) and M_1 is independent of t.

In the next two conditions $\hat{\omega} : \Omega \to \mathbb{R}_{>0}$ is a weight such that $L^p(\Omega,\hat{\omega}) \subset L^1(\Omega)$, with $1 < p < \infty$.

- **(e)** Given $g \in L^p(\Omega_t,\hat{\omega})$, with vanishing mean value, there exists a solution $\mathbf{v} \in W_0^p(\Omega_t,\hat{\omega})^n$ of $\operatorname{div}\mathbf{v} = g$ with

$$\|D\mathbf{v}\|_{L^p(\Omega_t,\hat{\omega})} \leq M_2\|g\|_{L^p(\Omega_t,\hat{\omega})},$$

  for all $t \in \Gamma$, where the positive constant M_2 does not depend on t.
- **(f)** The operator T defined in (3.5) is continuous from $L^p(\Omega,\hat{\omega})$ to itself with norm M_T.

An example of a collection of subdomains verifying **(e)** could be $\{\Omega_t\}_{t\in\Gamma}$ such that the constant C_{Ω_t} is uniformly bounded (for example, cubes) and the weight $\hat{\omega}$ satisfies that

$$\operatorname*{ess\,sup}_{x\in\Omega_t}\hat{\omega}(x) \leq C \operatorname*{ess\,inf}_{x\in\Omega_t}\hat{\omega}(x),$$

where C is independent of t.

Condition **(f)** is used to include the weight $\hat{\omega}$ in both sides of inequality (3.9). The case when $\hat{\omega} = 1$ was proved in general in Lemma 3.1.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\hat{\omega},\bar{\omega} : \Omega \to \mathbb{R}_{>0}$ two weights, with $L^p(\Omega,\hat{\omega}) \subset L^1(\Omega)$, for $1 < p < \infty$, and finally a collection $\{\Omega_t\}_{t\in\Gamma}$ of subdomains of Ω verifying conditions from* **(a)** *to* **(f)** *mentioned above. Hence, given $f \in L_0^p(\Omega,\hat{\omega})$ with vanishing mean value there exists a solution $\mathbf{u} \in W_0^{1,p}(\Omega,\bar{\omega}\hat{\omega})^n$ of $\operatorname{div}\mathbf{u} = f$ such that*

$$\|D\mathbf{u}\|_{L^p(\Omega,\bar{\omega}\hat{\omega})} \leq C\|f\|_{L^p(\Omega,\hat{\omega})}, \tag{3.9}$$

where

$$C = 2NM_1M_2\left(N + 2M_T^p\right)^{1/p}.$$

Proof. The collection of subdomains satisfies **(a)** and **(b)**, and from **(f)** the weight $\hat{\omega}$ makes the operator $T : L^p(\Omega, \hat{\omega}) \to L^p(\Omega, \hat{\omega})$ continuous. Thus, using Theorem 3.1 we can decompose the integrable function f as

$$f = \sum_{t\in\Gamma} f_t - \tilde{f}_t,$$

where $f_t - \tilde{f}_t \in L^p(\Omega_t, \hat{\omega})$, with vanishing mean value, and

$$\sum_{t\in\Gamma} \left(\operatorname*{ess\,inf}_{x\in\Omega_t} \omega(x) \right)^p \|f_t - \tilde{f}_t\|^p_{L^p(\Omega_t,\hat{\omega})} \le \sum_{t\in\Gamma} \|f_t - \tilde{f}_t\|^p_{L^p(\Omega_t,\omega\hat{\omega})} \le C_2 \|f\|^p_{L^p(\Omega,\hat{\omega})},$$

where $C_2 = 2^p \left(N + 2M_T^p \right)$.

Note that the essential infimum of ω over Ω_t is positive because of **(d)**, then $f_t - \tilde{f}_t$ belongs to $L^p(\Omega_t, \hat{\omega})$ as we announced. Now, using condition **(e)** there exists a solution $\mathbf{u}_t \in W_0^p(\Omega_t, \hat{\omega})^n$ of $\operatorname{div} \mathbf{u}_t = f_t - \tilde{f}_t$, with

$$\|D\mathbf{u}_t\|_{L^p(\Omega_t,\hat{\omega})} \le M_2 \|f_t - \tilde{f}_t\|_{L^p(\Omega_t,\hat{\omega})},$$

where M_2 is independent of t. Therefore, using requirement **(c)**, the vector field $\mathbf{u} := \sum_{t\in\Gamma} \mathbf{u}_t$ is a solution of $\operatorname{div} \mathbf{u} = f$. Moreover, using **(d)**

$$\begin{aligned}
\|D\mathbf{u}\|^p_{L^p(\Omega,\bar{\omega}\hat{\omega})} &\le N^p \sum_{t\in\Gamma} \|D\mathbf{u}_t\|^p_{L^p(\Omega_t,\bar{\omega}\hat{\omega})} \le N^p M_1^p \sum_{t\in\Gamma} \left(\operatorname*{ess\,inf}_{x\in\Omega_t} \omega(x) \right)^p \|D\mathbf{u}_t\|^p_{L^p(\Omega_t,\hat{\omega})} \\
&\le N^p M_1^p M_2^p \sum_{t\in\Gamma} \left(\operatorname*{ess\,inf}_{x\in\Omega_t} \omega(x) \right)^p \|f_t - \tilde{f}_t\|^p_{L^p(\Omega_t,\hat{\omega})} \\
&\le N^p M_1^p M_2^p 2^p \left(N + 2M_T^p \right) \|f\|^p_{L^p(\Omega,\hat{\omega})},
\end{aligned}$$

proving that $\mathbf{u}$ belongs to $W^{1,p}(\Omega, \bar{\omega}\hat{\omega})^n$ and the estimate claimed in the theorem.

Finally, let us prove that $\mathbf{u}$ belongs to $\overline{C_0^\infty(\Omega)^n}$. Given $\epsilon > 0$, using that $\sum_{t\in\Gamma} \|D\mathbf{u}_t\|^p_{L^p(\Omega_t,\bar{\omega}\hat{\omega})} < \infty$, there exists a finite set $\Gamma_0 \subset \Gamma$ such that

$$\sum_{t\in\Gamma\setminus\Gamma_0} \|D\mathbf{u}_t\|^p_{L^p(\Omega_t,\bar{\omega}\hat{\omega})} < N^{-p} \frac{\epsilon}{2}.$$

Now, for $t \in \Gamma_0$, we take $\mathbf{v}_t \in C_0^\infty(\Omega_t)^n$ such that

$$\|D\mathbf{u}_t - D\mathbf{v}_t\|^p_{L^p(\Omega_t,\hat{\omega})} \le N^{-p} M_1^{-p} \left(\operatorname*{ess\,inf}_{x\in\Omega_t} \omega(x) \right)^{-p} \frac{\epsilon}{2m},$$

where m is the cardinal of Γ_0. Thus, using that each Ω_t is included in Ω and Γ_0 is finite, $\mathbf{v} := \sum_{t\in\Gamma_0} \mathbf{v}_t$ belongs to $C_0^\infty(\Omega)^n$ and

$$\begin{aligned}
&\|D\mathbf{u} - D\mathbf{v}\|^p_{L^p(\Omega,\bar{\omega}\hat{\omega})} \\
\le\ & N^p \sum_{t\in\Gamma\setminus\Gamma_0} \|D\mathbf{u}_t\|^p_{L^p(\Omega_t,\bar{\omega}\hat{\omega})} + N^p M_1^p \sum_{t\in\Gamma_0} \left(\operatorname*{ess\,inf}_{x\in\Omega_t} \omega(x) \right)^p \|D\mathbf{u}_t - D\mathbf{v}_t\|^p_{L^p(\Omega_t,\hat{\omega})} < \epsilon,
\end{aligned}$$

completing the proof. □

In the next corollary we prove that Ω satisfies $(\mathrm{div})_p$, with an estimate of the constant C_Ω, if it is possible to decompose Ω by a good enough collection of subdomains $\{\Omega_t\}_{t\in\Gamma}$.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for which there exists a decomposition $\{\Omega_t\}_{t\in\Gamma}$ that fulfills* **(a)**, **(b)**, **(c)** *and* **(e)** *for $\hat{\omega} = 1$ and $1 < p < \infty$ such that $\omega(x) \ge \frac{1}{M_1}$ for all $x \in \Omega$. Hence, Ω satisfies $(div)_p$ and its constant C_Ω is bounded by*

$$C_\Omega \le 2M_1 M_2 N^{1+1/p} \left(1 + \frac{2^{p+1} p}{p-1} \right)^{1/p}.$$

Proof. This result is a consequence of the previous theorem using $\hat{\omega} = \bar{\omega} = 1$ and Lemma 3.1. $\square$

4. Divergence problem on general domains

In this section, we show the existence of a solution in a weighted Sobolev space of the divergence problem, $\operatorname{div} \mathbf{u} = f$, on an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$. The constant involved in the estimation of the solution is explicit and depends only on n and p. Furthermore, we use Whitney cubes to decompose the domain Ω, and the weight ω that we obtain for this decomposition depends locally on the ratio $\frac{|Q|}{|S(Q)|}$, where Q is a Whitney cube and $S(Q)$ is its shadow (defined in (4.11)). This type of ratios has been studied on some domains, for instance domains whose quasi-hyperbolic metric satisfies a logarithmic growth condition. See [17], [15] and [16] for more details.

In [10], the authors prove a similar result also for arbitrary bounded domains using an atomic decomposition obtained from a weighted Poincaré inequality, where the weight is related to the Euclidean geodesic distance in Ω.

Let $\mathcal{W} := \{Q_t\}_{t\in\Gamma}$ be a Whitney decomposition, i.e. a family of closed dyadic cubes whose interiors are pairwise disjoints, which satisfies

(i) $\Omega = \bigcup_{t\in\Gamma} Q_t$,
(ii) $diam(Q_t) \leq dist(Q_t, \partial\Omega) \leq 4\, diam(Q_t)$,
(iii) $\frac{1}{4}\, diam(Q_s) \leq diam(Q_t) \leq 4\, diam(Q_s)$, when $Q_s \cap Q_t \neq \emptyset$,

where $diam(Q)$ denotes the diameter of Q. Moreover, given a constant $\varepsilon \in (0, 1/4)$ which is arbitrary but will be kept fixed in what follows, Q_t^* denotes the open cube which has the same center as Q_t but is expanded by the factor $1 + \varepsilon$. This collection of expanded cubes satisfies that

$$\sum_{t\in\Gamma} \chi_{Q_t^*}(x) \leq 12^n \chi_\Omega, \text{ for all } x \in \mathbb{R}^n. \tag{4.10}$$

Moreover, Q_s^* intersects Q_t if and only if Q_s touches Q_t. See [20] for details.

Now, let us take a Whitney cube Q_a which will be distinguished from the rest. Then, for each Q_t in $\mathcal{W}$ we take a unique chain of cubes $\Omega_a = \Omega_{t_0}, \Omega_{t_1}, \cdots, \Omega_{t_k} = \Omega_t$ connecting Q_a with Q_t, such that for each $1 \leq i \leq k$ the intersection between Q_{t_i} and $Q_{t_{i-1}}$ is a $n-1$ dimensional face of one of those cubes. In addition, we assume that k is minimal over this type of chains and, using an inductive argument, that $\Omega_a, \Omega_{t_1}, \cdots, \Omega_{t_i}$ is the chain taken for each Ω_{t_i}, with $1 \leq i \leq k$.

Observe that using these chains connecting any $Q_t \in \mathcal{W}$ with Q_a in a unique way it is possible to define a rooted tree structure over Γ. Indeed, we say that two vertices $s, s' \in \Gamma$ are connected by an edge if and only if Q_s and $Q_{s'}$ are consecutive cubes in a chain $Q_a, Q_{t_1}, \cdots, Q_{t_k} = Q_t$, for some Q_t. As it is expected, the root of Γ is the index a of the distinguished cube Q_a. In addition, a partial order $\preceq$ over Γ is inherited.

Now, we define the shadow $S(Q_t)$ of a cube $Q_t \in \mathcal{W}$ as

$$S(Q_t) := \bigcup_{s\succeq t} Q_s^*. \tag{4.11}$$

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain, and $1 < p < \infty$. Given $f \in L^p_0(\Omega)$ there exists a vector field $\mathbf{u} \in W_0^{1,p}(\Omega, \bar{\omega})^n$ solution of $\operatorname{div} \mathbf{u} = f$ with the estimate*

$$\|D\mathbf{u}\|_{L^p(\Omega,\bar{\omega})} \leq C_{p,n} \|f\|_{L^p(\Omega)}, \tag{4.12}$$

where $C_{p,n}$ depends only on n and p, and $\bar{\omega}$ is defined over the interior of Q_s as

$$\bar{\omega} := \min_{Q_k \cap Q_s \neq \emptyset} \frac{|Q_k|}{|S(Q_k)|}.$$

The expanded cubes Q_s^* use in their definition $\varepsilon := 2^{-7}$.

Proof. Let us observe that $\bar{\omega}$ is defined almost everywhere but it is sufficient in this context.

This result is a consequence of Theorem 3.2. Let us consider the decomposition $\{\Omega_t\}_{t\in\Gamma}$ defined by $\Omega_t := Q_t^*$ with $\varepsilon := 2^{-7}$. From (4.10), it follows that the overlapping of the subdomains of this collection is bounded by $N = 12^n$.

Now, let us define the collection $\{B_t\}_{a\neq t\in\Gamma}$. Given t in $\Gamma\setminus\{a\}$ and its parent t_p, the intersection $Q_t\cap Q_{t_p}$ is a $n-1$ dimensional face of one of those. Let us denote by c^t the center of that $n-1$ dimensional cube and observe that the length of its sides is greater than $l_t/4$, where l_t is the length of the sides of Q_t. Thus, if we define the distance $d_\infty(x,y) := \max_{1\leq i\leq n}|x_i - y_i|$ it follows that $d_\infty(c_t, Q_s) \geq l_t/8$ for any $s\neq t, t_p$. Now, we define B_t as an open cube with center in c^t and sides with length l small enough in order to have the cube B_t included in $Q_t^*\cap Q_{t_p}^*$ and disjoint from Q_s^* for all $s\neq t, t_p$. It is known that $l_{t_p}\geq l_t/4$, then taking $l = \varepsilon l_t/4$ it follows that $B_t\subset Q_t^*\cap Q_{t_p}^*$. In particular, as $\varepsilon<1$ it can be seen that $B_t\subset Q_t\cup Q_{t_p}$. Hence, using that Q_s^* intersects Q_k if and only if Q_s touches Q_k, we can assert that if Q_s^* intersects $Q_t\cup Q_{t_p}$ then the common length of the sides of Q_s is lesser than $16 l_t$. Thus, $d_\infty(Q_s^*, c^t)\geq l_t/8 - 16 l_t\varepsilon/2$. Thus, it is sufficient to show that ε verifies that $l_t/8 - 16l_t\varepsilon/2 > \varepsilon l_t/8$. Hence, $B_t\subset\Omega_t\cap\Omega_{t_p}$ and $B_t\cap\Omega_s = \emptyset$ if $s\neq t, t_p$, obtaining a collection $\{B_t\}_{t\neq a}$ pairwise disjoint. Thus, the collection $\{\Omega_t\}_{t\in\Gamma}$ of subdomains of Ω verifies conditions **(a)** and **(b)** in Subsection 3.1.

In order to prove condition **(c)** on page 7, we can see that for any $x\in\Omega$ the ball with center x and radius $\frac{1}{2}d(x,\partial\Omega)$ intersects only a finite number of Q_s^*'s. Condition **(e)** is obtained by observing that $\hat{\omega}=1$ and the subdomains in the decomposition are cubes thus the constants C_{Ω_t} are equal to each other (see Lemma 2.1). Finally, condition **(f)** has been proved in Lemma 3.1, thus it only remains to prove **(d)**.

Now, given $t\in\Gamma$, it can be observed that $\omega(x)\neq 1$ over Q_t only if x belongs to B_s with $s=t$ or $s_p=t$. Thus, given $x\in Q_t$ it follows that

$$\omega(x) = \begin{cases} \dfrac{|B_t|}{|S(Q_t)|} = \dfrac{2^{-9n}|Q_t|}{|S(Q_t)|} & \text{if } x\in B_t \\[2ex] \dfrac{|B_s|}{|S(Q_s)|} = \dfrac{2^{-9n}|Q_s|}{|S(Q_s)|} \geq \dfrac{2^{-11n}|Q_t|}{|S(Q_t)|} & \text{if } x\in B_s \text{ with } s_p = t \\[2ex] 1 & \text{otherwise.} \end{cases}$$

Hence, using that Ω_t is included in $\displaystyle\bigcup_{Q_s\cap Q_t\neq\emptyset} Q_s$, it follows that

$$\operatorname*{ess\,sup}_{x\in\Omega_t}\bar{\omega}(x) = \max_{Q_s\cap Q_t\neq\emptyset}\operatorname*{ess\,sup}_{x\in Q_s}\bar{\omega}(x) \leq \frac{|Q_t|}{|S(Q_t)|} \leq 2^{11n}\operatorname*{ess\,inf}_{x\in\Omega_t}\omega(x),$$

obtaining condition **(e)** with $M_1 = 2^{11n}$. Thus, using Theorem 3.2 we obtain (4.12) with

$$C_{p,n} = 2C_Q 2^{11n} 12^{n+n/p}\left(1+\frac{2^{p+1}p}{p-1}\right)^{1/p},$$

where C_Q is the constant in (1.1) for an arbitrary cube Q.

□

5. Divergence problem and Stokes equations on domains with an external cusp

In this section we show the second application of our decomposition to prove the existence of a weighted solution of $\operatorname{div}\mathbf{u} = f$ on a class of n dimensional domains with an external cusp arbitrarily narrow. Similar domains were studied in [9] where the cusp is defined by a power function x^γ, with $\gamma > 1$.

Given a Lipschitz function $\varphi : [0,a] \to \mathbb{R}$ that satisfies the properties:

(i) $\varphi(0) = 0$, and $\varphi(r) > 0$ if $x \in (0,a]$,
(ii) $\varphi'(0) = 0$, $|\varphi'| \leq K_1$,
(iii) $\frac{\varphi(t)}{t} \leq K_2 \frac{\varphi(r)}{r}$, for all $0 < t < r \leq a$,

we define the following domain with a cusp at the origin:

$$\Omega_\varphi := \{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} \, : \, 0 < x < a \text{ and } |y| < \varphi(x)\} \subset \mathbb{R}^n. \tag{5.13}$$

The following are three examples of functions which verify (i), (ii) and (iii):

- $\varphi(x) = x^\gamma$, with $\gamma > 1$.
- $\varphi(x) = e^{-1/x^2}$ in $(0,a]$ and $\varphi(0) = 0$.
- $\varphi(x) = x^\gamma(2 + \sin(x^{1-\gamma}))$ in $(0,a]$, with $\gamma > 1$, and $\varphi(0) = 0$.

The Lipschitz condition in (ii) keeps $\partial\Omega_\varphi$ from having cusps different from the one at the origin, and condition (iii) is used to solve a technical issue when κ in Theorem 5.1 is positive.

Let us start introducing a decreasing sequence in the interval $(0,a]$ and some of its properties. This sequence will be used to define a decomposition of Ω_φ. Thus, we define inductively a decreasing sequence $\{x_i\}_{i\geq 0}$ in $(0,a]$ with $x_0 = a$, and x_{i+1} the maximum number in $(0,x_i)$ satisfying that $\varphi(x) = x_i - x$. The well definition of this sequence is based on the continuity of $\varphi(x) + x$, which satisfies that $\varphi(0) + 0 < x_i$ and $\varphi(x_i) + x_i > x_i$, using that φ is continuous and positive on $(0,a]$, with $\varphi(0) = 0$. In addition, it can be seen that $\{x_i\}_{i\geq 0}$ decreases to 0. Indeed, if $\{x_i\}_{i\geq 0}$ converges to $\bar{x} \geq 0$, then

$$\varphi(\bar{x}) = \lim_{i\to\infty} \varphi(x_{i+1}) = \lim_{i\to\infty} x_i - x_{i+1} = 0.$$

Hence, $\bar{x} = 0$.

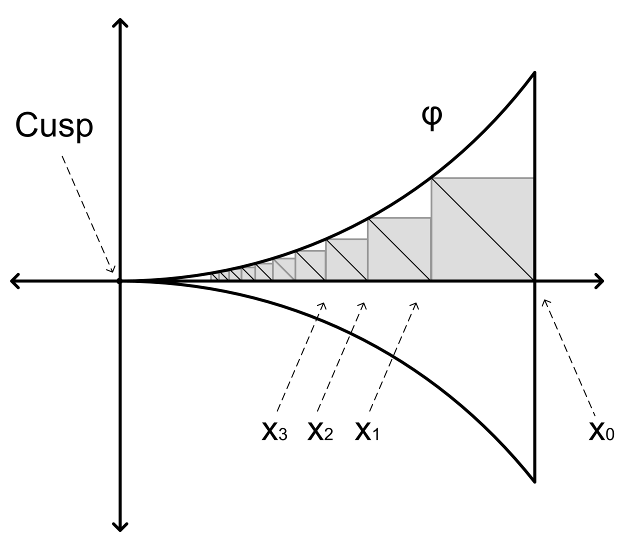

Figure 1. A simple example with an increasing φ

Let us see some properties of this sequence. Taking $x_{i+1} \leq x \leq x_i$, we can assert that

$$x_{i+1} \leq x \leq (K_1 + 1)x_{i+1} \tag{5.14}$$

and

$$\frac{1}{K_2}\varphi(x_{i+1}) \leq \varphi(x) \leq (K_1+1)\varphi(x_{i+1}). \tag{5.15}$$

The first inequality in (5.14) holds by definition and the second one can be proved by observing that

$$x \leq x_i = \varphi(x_{i+1}) + x_{i+1} = \frac{\varphi(x_{i+1}) - \varphi(0)}{x_{i+1} - 0} x_{i+1} + x_{i+1}.$$

The second inequality in (5.15) follows from

$$|\varphi(x) - \varphi(x_{i+1})| \leq K_1(x - x_{i+1}) \leq K_1(x_i - x_{i+1}) = K_1\varphi(x_{i+1}),$$

and the first one can be proved by

$$\varphi(x_{i+1}) \leq K_2\varphi(x)\,\frac{x_{i+1}}{x} \leq K_2\varphi(x).$$

Now, we introduce the collection $\{\Omega_i\}_{i\in\mathbb{N}_0}$ of subdomains of Ω_φ to be used with Theorem 3.2. Indeed, given $i \in \mathbb{N}_0$ we define

$$\Omega_i := \{(x,y) \in \Omega_\varphi \,:\, x_{i+2} < x < x_i\}. \tag{5.16}$$

Note that in this case $\Gamma = \mathbb{N}_0$ where two vertices i and j are connected by an edge if and only if $|i-j| = 1$. Moreover, if we take the root $a = 0$, the partial order $\preceq$ inherited from this tree structure coincides with the total order $\leq$ of $\mathbb{N}_0$.

The following theorem is the main result of this section.

Theorem 5.1 (Divergence on cuspidal domains)**.** *Let $\Omega_\varphi \subset \mathbb{R}^n$ be the domain defined in (5.13), $1 < p < \infty$ and $\kappa \geq 0$. Given $f \in L^p(\Omega, \varpi^{-\kappa})$, with vanishing mean value, there exists a solution $\mathbf{u}$ in $W^{1,p}_0(\Omega_\varphi, \varpi^{1-\kappa})^n$ of $\operatorname{div}\mathbf{u} = f$ satisfying that*

$$\|D\mathbf{u}\|_{L^p(\Omega_\varphi,\varpi^{1-\kappa})} \leq C\|f\|_{L^p(\Omega_\varphi,\varpi^{-\kappa})}, \tag{5.17}$$

where $\varpi(x,y) = \dfrac{\varphi(x)}{x}$, and C depends only on K_1, K_2, p, n and κ.

Proof. As we mentioned before this theorem is an application of Theorem 3.2. The collection of subdomains $\{\Omega_i\}_{i\in\mathbb{N}_0}$ has been defined in (5.16), and the weights $\hat{\omega}, \bar{\omega} : \Omega_\varphi \to \mathbb{R}_{>0}$ are defined by $\bar{\omega} := \varpi$ and $\hat{\omega} := \varpi^{-\kappa}$. Observe that Ω_i is indeed a subdomain of Ω_φ. Moreover, $\varpi \leq K_1$ and $\kappa \geq 0$, thus $L^p(\Omega_\varphi, \varpi^{-\kappa}) \subset L^1(\Omega_\varphi)$. Then, it just remains to prove conditions **(a)** to **(f)** on pages 4 and 7.

Just from the definition of the collections of subdomains it can be observed that conditions **(a)** and **(c)** hold, with a constant $N = 2$. The collection $\{B_i\}_{i\geq 1}$ defined below verifies **(b)**:

$$B_i := \Omega_i \cap \Omega_{i-1} = \{(x,y) \in \Omega_\varphi \,:\, x_{i+1} < x < x_i\}.$$

Now, let us prove **(d)**. The weight ω defined in (3.6) is equal to $\omega(x,y) = \frac{|B_i|}{|W_i|}$ over B_i, where

$$W_i \;=\; \bigcup_{k\geq i} \Omega_k = \{(x,y) \in \Omega_\varphi \,:\, x < x_i\}.$$

Thus, given $(x,y) \in B_i$, for $i \geq 1$, using inequalities (5.14) and (5.15), and (iii), it follows that

$$\omega(x,y) = \frac{|B_i|}{|W_i|} \geq \frac{C_{K_2,n}\,\varphi(x_{i+1})^n}{C_{K_2,n}\,x_i\varphi(x_i)^{n-1}} \geq C_{K_1,K_2,n}\frac{\varphi(x_i)}{x_i}.$$

Now, given $(x,y) \in B_{i+1}$, and using that $\frac{\varphi(x_{i+1})}{x_{i+1}} \geq \frac{1}{K_1+1}\frac{\varphi(x_i)}{x_i}$ obtained from (5.14) and (5.15), we can conclude that

$$\omega(x,y) = \frac{|B_{i+1}|}{|W_{i+1}|} \geq C_{K_1,K_2,n}\frac{\varphi(x_{i+1})}{x_{i+1}} \geq C_{K_1,K_2,n}\frac{\varphi(x_i)}{x_i}.$$

Hence, using (iii) and the previous inequalities,

$$\operatorname*{ess\,sup}_{x\in\Omega_i} \varpi(x) \leq K_2\frac{\varphi(x_i)}{x_i} \leq C_{K_1,K_2,n} \operatorname*{ess\,inf}_{x\in\Omega_i} \omega(x).$$

Now, let us prove **(f)**, the continuity of the operator T and an estimation of its norm. In order to simplify the notation we denote $\varpi(x)$ instead of $\varpi(x,y)$. The proof uses (iii), the fact that $\kappa \geq 0$, and the continuity of T without weight shown in Lemma 3.1. Indeed,

$$\begin{aligned}
\int_{\Omega_\varphi} |Tg(x,y)|^p \varpi^{-p\kappa}(x) &= \sum_{i\geq 1}\int_{B_i} \varpi^{-p\kappa}(x)\left(\frac{1}{|W_i|}\int_{W_i}|g|\varpi^{-\kappa}\varpi^{\kappa}\right)^p \\
&\leq \sum_{i\geq 1}\int_{B_i} \varpi^{-p\kappa}(x)K_2^{p\kappa}\varpi^{p\kappa}(x_i)\left(\frac{1}{|W_i|}\int_{W_i}|g|\varpi^{-\kappa}\right)^p \\
&\leq K_2^{2p\kappa}\sum_{i\geq 1}\left(\frac{\varpi(x_i)}{\varpi(x_{i+1})}\right)^{p\kappa}\int_{B_i}\left(\frac{1}{|W_i|}\int_{W_i}|g|\varpi^{-\kappa}\right)^p \\
&\leq C\sum_{i\geq 1}\int_{B_i}\left(\frac{1}{|W_i|}\int_{W_i}|g|\varpi^{-\kappa}\right)^p \\
&\leq C\|g\varpi^{-\kappa}\|^p_{L^p(\Omega_\varphi)} = C\|g\|^p_{L^p(\Omega_\varphi,\varpi^{-\kappa})},
\end{aligned}$$

where C depends only on K_1, K_2, p and κ.

Finally, it just remains to prove **(e)**. Using (iii), (5.14) and (5.15), we obtain that

$$\frac{1}{(K_1+1)^2}\frac{\varphi(x_i)}{x_i} \leq \frac{\varphi(x)}{x} \leq K_2\frac{\varphi(x_i)}{x_i},$$

for all $x_{i+2} \leq x \leq x_i$. Thus, we can assume that $\varpi^{-\kappa}$ is the constant function with value $\left(\frac{\varphi(x_i)}{x_i}\right)^{-\kappa}$ over Ω_i. Thus, it is enough to prove **(e)** when $\hat{\omega} = 1$. The proof of this case is shown in Lemma 5.1. □

The next result follows immediately from Theorem 5.1.

Theorem 5.2 (Stokes on cuspidal domains)**.** *Given $\Omega_\varphi \subset \mathbb{R}^n$ the domain defined in (5.13), $h \in L^2_0(\Omega_\varphi, \varpi^{-1})$, and $\mathbf{g} \in H^{-1}(\Omega_\varphi)^n$. There exists a unique solution $(\mathbf{u}, p) \in H^1_0(\Omega_\varphi)^n \times L^2(\Omega_\varphi, \varpi)$ of (2.2), with $\int_{\Omega_\varphi} p\varpi^2 = 0$. Moreover,*

$$\|D\mathbf{u}\|_{L^2(\Omega_\varphi)} + \|p\|_{L^2(\Omega_\varphi,\varpi)} \leq C\left(\|\mathbf{g}\|_{H^{-1}(\Omega_\varphi)} + \|h\|_{L^2(\Omega_\varphi,\varpi^{-1})}\right),$$

where $\varpi(x,y) = \dfrac{\varphi(x)}{x}$, and C depends only on Ω_φ.

Proof. By Theorem 5.1, there exists $\tilde{\mathbf{v}} \in H^1_0(\Omega_\varphi)^n$ satisfying that $\operatorname{div}\tilde{\mathbf{v}} = h$ and the estimate (5.17). Then, $\Delta\tilde{\mathbf{v}} \in H^{-1}(\Omega_\varphi)^n$. Now, using the Theorem stated on page 3, and Theorem 5.1, we can conclude that there exists a unique solution $(\mathbf{v}, p) \in H^1_0(\Omega_\varphi)^n \times L^2(\Omega_\varphi, \varpi)$ of

$$\begin{cases} -\Delta\mathbf{v} + \nabla p &= \mathbf{g} + \Delta\tilde{\mathbf{v}} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega, \end{cases} \tag{5.18}$$

with $\int_{\Omega_\varphi} p\varpi^2 = 0$, and

$$\begin{aligned} \|D\mathbf{v}\|_{L^2(\Omega_\varphi)} + \|p\|_{L^2(\Omega_\varphi,\varpi)} &\leq C\left(\|\mathbf{g}\|_{H^{-1}(\Omega_\varphi)} + \|\Delta\tilde{\mathbf{v}}\|_{H^{-1}(\Omega_\varphi)}\right) \\ &\leq C\left(\|\mathbf{g}\|_{H^{-1}(\Omega_\varphi)} + \|f\|_{L^2(\Omega_\varphi,\varpi)}\right). \end{aligned}$$

Finally, $(\mathbf{u}, p) = (\mathbf{v} + \tilde{\mathbf{v}}, p)$ is the solution mentioned in the theorem. □

Next, we prove a lemma used in the proof of Theorem 5.1.

Lemma 5.1. *Let $\Omega_i \subset \mathbb{R}^n$ be, with $i \geq 0$, the domain defined on (5.16), and $1 < p < \infty$. Then, Ω_i satisfies $(div)_p$ and there exists a constant C depending only on K_1, K_2, n, and p such that*

$$C_{\Omega_i} \leq C$$

for all $i \geq 0$, where C_{Ω_i} is the constant in (1.1).

Proof. Ω_i is a Lipschitz domain, and it is well known that a domain of this type satisfies $(\mathrm{div})_p$. What this lemma states is the existence of an uniform upper bound of C_{Ω_i}. The idea of the proof of this result is to show that each Ω_i can be written as the finite union of certain star-shaped domains with respect to a ball (for which there exist estimates of the constants) and then to apply Corollary 3.1. The number of domains in the union does not depend on i.

Let us recall the definition of this class of domains. A domain U is **star-shaped** with respect to a ball B if and only if any segment with an end-point in U and the other one in B is contained in U. This class of domains has the following estimate of the constant on the divergence problem (1.1): if R denotes the diameter of U and ρ the radius of the ball $B \subset U$, the constant C_U is bounded by

$$C_U \leq C_{n,p}\left(\frac{R}{\rho}\right)^{n+1}. \tag{5.19}$$

See Lemma III.3.1 in [13] for details.

Let us show the way to split Ω_i into the finite union. Without loss of generality, we assume that $K_1, K_2 \geq 1$. Thus, from (5.14) and (5.15) it follows that

$$\varphi(x_{i+1}) \leq |x_i - x_{i+2}| \leq 2K_2\varphi(x_{i+1}), \tag{5.20}$$

and

$$\frac{1}{2K_1K_2}\varphi(x_{i+1}) \leq \varphi(x) \leq 2K_1K_2\varphi(x_{i+1}), \tag{5.21}$$

if x belongs to $[x_{i+2}, x_i]$. Next, we take a natural number m such that $m > 16K_1^2K_2^2$ (this number m could be arbitrarily big but it is fixed) and the $m+1$ equidistant points $r_0 < r_1 < \cdots < r_m$, with $r_0 = x_{i+2}$ and $r_m = x_i$. Thus, for $1 \leq j \leq m$, it follows that

$$\frac{1}{m}\varphi(x_{i+1}) \leq |r_j - r_{j-1}| \leq \frac{1}{8K_1^2K_2}\varphi(x_{i+1}). \tag{5.22}$$

Thus, the collection of star-shaped subdomains of Ω_i to be considered is $\{U_1, \cdots, U_{m-1}\}$ where

$$U_j := \{(x, y) \in \Omega_i \,:\, r_{j-1} < x < r_{j+1}\},$$

for $1 \leq j \leq m-1$. The tree structure of the index set $\{1, \cdots, m-1\}$ is the same that we have defined on $\mathbb{N}_0$, where $a = 1$ is the root in this case. Moreover, we introduce the collection $\{B_j\}_{2\leq j\leq m-1}$ as

$$B_j := U_j \cap U_{j-1} = \{(x, y) \in \Omega_i \,:\, r_{j-1} < x < r_j\}.$$

Thus, let us prove the hypothesis of Corollary 3.1. Conditions **(a)**, **(b)** and **(c)** follow easily with $N = 2$. From (5.20), (5.21) and (5.22), the measure of any B_j and the whole Ω_i are comparable to $\varphi(x_{i+1})^n$. Thus, $\omega \geq \frac{1}{M_1}$, where M_1 depends only on K_1, K_2, n and m.

Finally, we just have to prove **(e)** for $\hat{\omega} = 1$. Let us show first that each U_j, for $1 \le j \le m-1$, is a star-shaped domain with respect to the ball $\hat{U}_j$ with center $(r_j, 0)$ and radius $(r_{j+1} - r_{j-1})/2$. Thus, given $(x,y) \in \hat{U}_j$, $(\tilde{x}, \tilde{y}) \in U_j$, and $s \in (0,1)$, we have to prove that

$$|sy + (1-s)\tilde{y}| < \varphi(sx + (1-s)\tilde{x}).$$

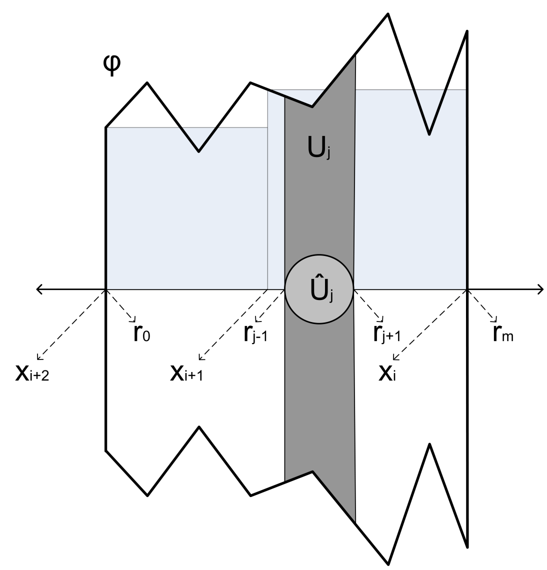

FIGURE 2. A more general φ and the star-shaped domain U_j

To simplify the notation we introduce $M := \varphi(x_{i+1})/2K_2K_1$. Thus,

$$\begin{aligned} |sy + (1-s)\tilde{y}| &< s\frac{M}{4K_1} + (1-s)\varphi(\tilde{x}) \le s\left(\frac{M}{4} - M\right) + \varphi(\tilde{x}) \\ &\le -\frac{3}{4}sM + (\varphi(\tilde{x}) - \varphi(sx + (1-s)\tilde{x})) + \varphi(sx + (1-s)\tilde{x}) \\ &\le -\frac{3}{4}sM + K_1 s|\tilde{x} - x| + \varphi(sx + (1-s)\tilde{x}) \\ &\le -\frac{3}{4}sM + K_1 s\frac{M}{2K_1} + \varphi(sx + (1-s)\tilde{x}) < \varphi(sx + (1-s)\tilde{x}). \end{aligned}$$

Now, using (5.21) and (5.22), it can be seen that the diameter of U_j and the radius of $\hat{U}_j$ are comparable to $\varphi(x_{i+1})$. Thus, using estimate (5.19), we can conclude that M_2 in **(e)** can be taken as a constant just depending on K_1, K_2, n and p. □

6. Divergence problem and Stokes equations on Hölder domains

In this section, we show the existence of a right inverse of the divergence operator and the well-posedness of the Stokes equations on an arbitrary bounded Hölder-α domain Ω, i.e. the boundary of Ω is locally the graph of a function that verifies $|\varphi(x) - \varphi(x')| \le K_\varphi |x - x'|^\alpha$, for all x, x'. Thus, we start this section studying a domain in $\mathbb{R}^n$ defined by the graph of a positive Hölder-α function $\varphi : (\frac{-3l}{2}, \frac{3l}{2})^{n-1} \to \mathbb{R}$, where $0 < \alpha \le 1$ and $l > 0$,

$$\Omega_\varphi := \left\{(x,y) \in (-l/2, l/2)^{n-1} \times \mathbb{R} \,:\, 0 < y < \varphi(x)\right\} \subset \mathbb{R}^n. \qquad (6.23)$$

In addition, we assume that $\varphi \ge 2l$ but $\varphi \not\ge 3l$, and $l \le 1$. Now, Ω is locally as Ω_φ, however the distance to the boundary of Ω is not necessarily equivalent to the distance to the graph of φ defined over $(-l/2, l/2)^{n-1}$. Thus, in order to solve this problem, we assume that Ω is locally an expanded version of Ω_φ:

$$\Omega_{\varphi,E} := \{(x, y) \in (-3l/2, 3l/2)^{n-1} \times \mathbb{R} : y < \varphi(x)\} \subset \mathbb{R}^n. \tag{6.24}$$

With this new approach of the problem, the distance to $\partial\Omega$ is equivalent to the distance to G over Ω_φ, where

$$G := \{(x, y) \in (-3l/2, 3l/2)^{n-1} \times \mathbb{R} : y = \varphi(x)\}. \tag{6.25}$$

Let us denote the distance to G as d_G.

Lemma 6.1. *Let Ω_φ be the domains defined on (6.23), $1 < p < \infty$, and $\kappa \geq 0$. Given $f \in L^p(\Omega_\varphi, d_G^{-\kappa})$ with vanishing mean value, there exists a vector field $\mathbf{u} \in W_0^{1,p}(\Omega_\varphi, d_G^{1-\alpha-\kappa})^n$ solution of $div\,\mathbf{u} = f$ such that*

$$\|D\mathbf{u}\|_{L^p(\Omega_\varphi, d_G^{1-\alpha-\kappa})} \leq C\|f\|_{L^p(\Omega_\varphi, d_G^{-\kappa})},$$

where C depends only on K_φ, α, n, p and κ.

Proof. We start defining a collection of cubes in the style of the Whitney cubes, but in this case the diameter of the cubes is comparable to d_G instead of the distance to $\partial\Omega_\varphi$. The construction of this collection of open cubes $\{Q_t\}_{t\in\Gamma}$ consists on piling cubes as boxes one over the other one in such a way that the common length of the sides of each cube is comparable to d_G. The cubes are constructed level by level starting by level 0, which has just the cube $Q_a = (\frac{-l}{2}, \frac{l}{2})^{n-1} \times (0, l)$. The construction of the cubes induces the tree structure of the index set Γ where the parents of the index of the cubes in level $m + 1$ are the index of the cubes in level m. Thus, suppose that we have defined all the cubes in level m, and let $Q_t = Q'_t \times (y_{t,1}, y_{t,2})$ be one of them. Let us denote by l_t the common length of the sides of Q_t. Thus, all the cubes Q_s's in level $m + 1$ with $s_p = t$ are defined in the following way: we move up and then expand Q_t to obtain $Q = 3(Q_t + (0, \cdots, 0, l_t))$ thus

(i) if $Q \subset \Omega_{\varphi,E}$ there is just one cube Q_s on level $m + 1$ such that $s_p = t$ and it is $Q_s = Q_t + (0, \cdots, 0, l_t)$,
(ii) if $Q \not\subset \Omega_{\varphi,E}$ there are 2^{n-1} cubes Q_s with $s_p = t$ and they are written as $Q_s = Q'_{t,1/2} \times (y_{t,2}, y_{t,2} + l_t/2)$, where $Q'_{t,1/2}$ is one of cubes in $\mathbb{R}^{n-1}$ obtained by splitting Q'_t into 2^{n-1} cubes with length of its sides equal to $l_t/2$.

See figure 3 for an example of the construction.

Note that the common length of the sides of the cubes $\{Q_t\}_{t\in\Gamma}$ decreases with respect to the order $\succeq$ inherited from the tree. Indeed,

$$l_t \leq l_s \qquad\qquad \text{if } t \succeq s. \tag{6.26}$$

Let us show another property satisfied by this collection. Given Q_t, it satisfies

$$l_t \leq dist(Q_t, G) \leq C_n\, l_t. \tag{6.27}$$

The first inequality follows from the definition. To prove the second one, note that there exists Q_s with $s \preceq t$ and $l_s = 2l_t$ such that $3(Q_s + (0, \cdots, 0, l_s)) \not\subset \Omega_{\varphi,E}$. Thus, following the notation $Q_s = Q'_s \times (y_{s,1}, y_{s,2})$ and $Q_t = Q'_t \times (y_{t,1}, y_{t,2})$, we can assert that there exists $x_s \in 3Q'_s$ such that $\varphi(x_s) < y_{s,2} + 2l_s \leq y_{t,1} + 4l_t$.

On the other hand, $\varphi \geq y_{t,1} + 2l_t$ in Q'_t, and $Q'_t \subseteq Q'_s$. Thus, using the convexity of $3Q'_s$ and the continuity of φ, there is another point in $3Q'_s$, let us represent it with the same notation x_s, such that

$$y_{t,1} + 2l_t \leq \varphi(x_s) \leq y_{t,1} + 4l_t. \tag{6.28}$$

Observe that any point in Q'_t has a distance to x_s less than the diameter of $3Q'_s$, which equals $6\sqrt{n-1}\, l_t$. Thus,

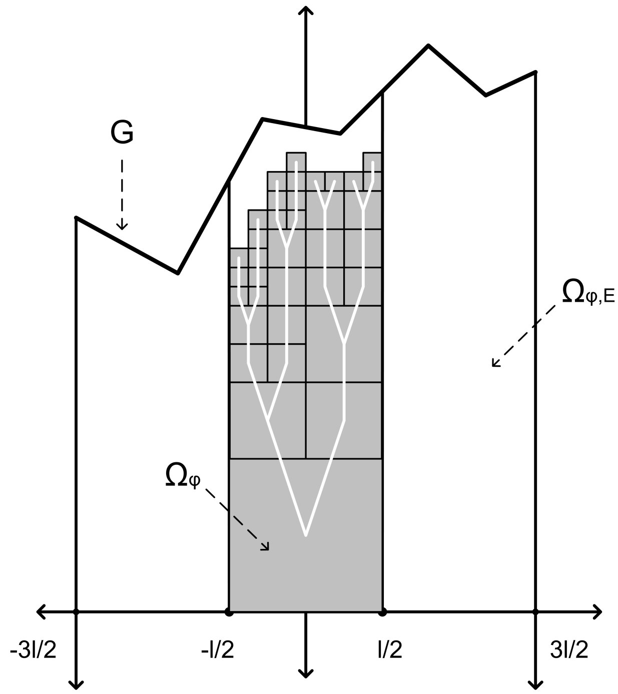

FIGURE 3. Under the graph of a Hölder-α function

$$dist(Q_t, G) \le dist(Q_t, (x_s, \varphi(x_s))) \le \sqrt{diam(3Q'_s)^2 + (3l_t)^2} \le 3\sqrt{4n-3}\, l_t.$$

Now, we are ready to define the collection $\{\Omega_t\}_{t\in\Gamma}$ of subdomains of Ω_φ to apply Theorem 3.2. The first subdomain Ω_a is the cube Q_a, and the other ones are the n dimensional rectangles defined by

$$\Omega_t := Q'_t \times (y_{t,1} - \frac{1}{2} l_t, y_{t,2}), \tag{6.29}$$

where $Q_t = Q'_t \times (y_{t,1}, y_{t,2})$. Observe that $y_{t,2} = y_{t,1} + l_t$.

It can be seen that conditions **(a)** and **(c)** are valid, with a constant $N = 2$. Indeed, $\Omega_t \cap \Omega_s \neq \emptyset$ for $t \neq s$ if and only if one index is the parent of the other one.

Next, if we define the collection $\{B_t\}_{t\neq a}$ as

$$B_t := \Omega_t \cap \Omega_{t_p} = Q'_t \times (y_{t,1} - \frac{1}{2} l_t, y_{t,1}),$$

(b) follows.

Now, let us show that **(d)** is verified with $\bar{\omega} := d_G^{1-\alpha}$. In order to estimate ω, we take the point $x_s \in \mathbb{R}^{n-1}$ that verifies (6.28). Then , for all $x_t \in Q'_t$ it follows that

$$|\varphi(x_t)| \le |\varphi(x_t) - \varphi(x_s)| + |\varphi(x_s)| \le C_{n,\alpha} K_\varphi l_t^\alpha + 4l_t + y_{t,1}.$$

Thus, $|W_t| \le C_{n,\alpha}(K_\varphi + l_t^{1-\alpha}) l_t^{n-1+\alpha}$ and $|B_t| = \frac{1}{2} l_t^n$. Then, given $s \in \Gamma$ with $s_p = t$, we have that $|W_s| \le |W_t|$ and $|B_s| \ge 2^{-n}|B_t|$. Thus,

$$\omega(x,y) \ge \frac{C_{n,\alpha}}{(K_\varphi + l^{n-1+\alpha})} l_t^{1-\alpha} = \frac{C_{n,\alpha}}{(K_\varphi + l^{n-1+\alpha})} d_G^{1-\alpha},$$

if (x, y) belong to Ω_t. Then,

$$\operatorname*{ess\,inf}_{(x,y)\in\Omega_t} \omega(x,y) \ge \frac{C_{n,\alpha}}{(K_\varphi + l^{n-1+\alpha})} l_t^{1-\alpha} \ge \operatorname*{ess\,sup}_{(x,y)\in\Omega_t} \frac{C_{n,\alpha}}{(K_\varphi + l^{n-1+\alpha})} d_G^{1-\alpha}(x,y),$$

proving **(d)** with a constant M_1 depending only on K_φ, n, α and l.

In order to study **(e)** we take $\hat{\omega} := d_G^{-\kappa}$. Using that $\kappa \geq 0$ and d_G is bounded over Ω_φ, it follows that $L^p(\Omega_\varphi, d_G^{-\kappa}) \subset L^1(\Omega_\varphi)$. Next, from (6.27) we have that $\frac{1}{2} l_t \leq dist(\Omega_t, G) \leq C_n\, l_t$, hence we can assume that $d_G^{1-\alpha}$ is constant over Ω_t reducing the problem to the case $\kappa = 0$. Now, Ω_t, with $t \neq a$, is a translate of $(0, l_t)^{n-1} \times (0, 3l_t/2)$, thus, using Lemma 2.1, we can assert that Ω_t satisfies $(\text{div})_p$ with a constant depending only on n.

Finally, condition **(f)** follows from (6.26) and Lemma 3.1. Moreover, it can be observed that $d_G \leq C_n l_s$ in Ω_s and $l_s \leq l_t$ if $s \succeq t$, hence $d_G \leq C_n l_t$ over W_t.

Now, given $g \in L^p(\Omega_\varphi, d_G^{-\kappa})$ the function $g d_G^{-\kappa}$ belongs to $L^p(\Omega_\varphi)$, and using Lemma 3.1, we have

$$\begin{aligned}
\int_\Omega |Tg|^p d_G^{-p\kappa} &= \sum_{t \neq a} \int_{B_t} d_G^{-p\kappa} \left(\frac{1}{|W_t|} \int_{W_t} |g| d_G^{-\kappa} d_G^{\kappa} \right)^p \\
&\leq C_{n,p,\kappa} \sum_{t \neq a} \int_{B_t} d_G^{-p\kappa} l_t^{p\kappa} \left(\frac{1}{|W_t|} \int_{W_t} |g| d_G^{-\kappa} \right)^p \\
&= C_{n,p,\kappa} \sum_{t \neq a} \int_{B_t} \left(\frac{1}{|W_t|} \int_{W_t} |g| d_G^{-\kappa} \right)^p \leq c_{n,p,\kappa} \| g d_G^{-\kappa} \|_{L^p(\Omega_\varphi)},
\end{aligned}$$

proving the lemma. □

Theorem 6.1 (Divergence on Hölder-α domains)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Hölder-α domain, and $1 < p < \infty$. Given $f \in L^p_0(\Omega, d^{-\kappa})$, with d the distance to $\partial\Omega$ and $\kappa \geq 0$, there exists a vector field $\mathbf{u} \in W^{1,p}_0(\Omega, d^{1-\alpha-\kappa})^n$ solution of $div\, \mathbf{u} = f$ with estimate*

$$\| D\mathbf{u} \|_{L^p(\Omega, d^{1-\alpha-\kappa})} \leq C \| f \|_{L^p(\Omega, d^{-\kappa})},$$

where C does not depend on f.

Proof. Ω is a Hölder-α domain, thus $\partial\Omega$ is locally the graph of a Hölder-α function, after taking a rigid movement. In fact, we can assume that $\partial\Omega$ can be covered by a finite collection of open sets $\{U_i\}_{1 \leq i \leq m}$ such that $\Omega_i := U_i \cap \Omega$ is in the form (6.23), where the extended domain in (6.24) is the intersection of another open set $V_i \supset U_i$ with Ω. The reason to consider these V_i's is just to have d locally comparable to d_G. Also, it can be assumed that the finite collection $\{\Omega_i\}$ is minimal in the sense that for each $1 \leq i \leq m$ the set $\Omega_i \setminus \bigcup_{j \neq i} U_j$ has a positive Lebesgue measure. Now, let us take a Lipschitz domain $\Omega_0 \subset\subset \Omega$ such that $B_i := (\Omega_i \cap U_0) \setminus \bigcup_{j \neq i} U_j$ has a positive Lebesgue measure and $\cup_{i=0}^m \Omega_i = \Omega$.

Let us define the finite collection $\{\Omega_i\}_{0 \leq i \leq m}$. The tree structure of the index set $\Gamma = \{0, 1, \cdots, m\}$ is defined in such a way that two nodes i and j are connected by an edge if and only if one of those is the root $a = 0$. Thus, the partial order is given by $i \preceq j$ if and only if $i = 0$.

The proof of this theorem follows the idea used to prove Theorem 3.2 with a minor difference in condition **(e)**. In this case, the condition has two different weights and it can be stated as: given $g \in L^p(\Omega_t, d^{-\kappa})$ with vanishing mean value there is a solution $\mathbf{v} \in W^p_0(\Omega_t, d^{1-\alpha-\kappa})^n$ of $\text{div}\, \mathbf{v} = g$ with

$$\| D\mathbf{v} \|_{L^p(\Omega_t, d^{-\kappa})} \leq M_2 \| g \|_{L^p(\Omega_t, d^{1-\alpha-\kappa})}$$

for all $t \in \Gamma$, where the positive constant M_2 does not depend on t. This condition was proved in Lemma 6.1.

Now, note that ω takes finite different positive values, thus the weight $\bar{\omega}$ can be assumed constant over Ω obtaining **(d)**. On the other hand, the operator T has its support in Ω_0, which is compactly contained in Ω. Thus, the weight $\hat{\omega} := d^{1-\alpha-\kappa}$ can be assumed constant. Hence, from Lemma 3.1, the operator $T : L^p(\Omega, d^{1-\alpha-\kappa}) \to L^p(\Omega, d^{1-\alpha-\kappa})$ is continuous and **(f)** is satisfied.

It can be noted that condition **(a)**, **(b)** and **(c)** can be easily verified from the definition of the collection of subdomains and it finiteness. Thus, the proof goes as in Theorem 3.2. $\square$

In the last theorem we show the well-posedness of the Stokes equations on bounded Hölder-α domains.

Theorem 6.2 (Stokes on Hölder-α domains)**.** *Given $\Omega \subset \mathbb{R}^n$ a bounded Hölder-α domain, and two functions $h \in L^2_0(\Omega, d^{\alpha-1})$, and $\mathbf{g} \in H^{-1}(\Omega)^n$. Then, there exists a unique solution $(\mathbf{u}, p) \in H^1_0(\Omega)^n \times L^2(\Omega, d^{1-\alpha})$ of (2.2) with $\int_\Omega p d^{2(1-\alpha)} = 0$. Moreover,*

$$\|D\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega, d^{1-\alpha})} \leq C \left(\|\mathbf{g}\|_{H^{-1}(\Omega)} + \|h\|_{L^2(\Omega, d^{\alpha-1})} \right), \tag{6.30}$$

where d is the distance to $\partial\Omega$, and C depends only on Ω.

Proof. The proof of this theorem follows the same idea as the one in the proof of Theorem 5.2. $\square$

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